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STUDIES IN CLASSICAL ELECTRON THEORY I

Radiation Damping and Differential Conservation Laws

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Synopsis

The present study is the first in a series of three attempting a critical assessment of the status of classical electron theory.

By means of a simple idealized experiment the relationship is exhibited between the retardation of physical actions and the requirement of energy conservation on the one hand and the occurrence of radiation damping on the other. From this example it emerges, that radiation phenomena are characterized by a certain feature of wholeness, even within the domain of classical physics.

The mechanism of radiation reaction is further analysed within the context of the classical Maxwell theory, where the phenomenon of damping – like that of electromagnetic inertia – naturally originates in the mutual interaction between the various infinitesimal constituents making up any finite change. On this background the interplay between the various assumptions, underlying the attempts initiated by Dirac of incorporating the notion of an "ideal point charge" into the foundation of classical electrodynamics, is critically examined.

In the "point electron description" the phenomenon of damping has no natural place, although the proponents of this description have offered several arguments leading to the well-known expression for the damping. Closer scrutiny reveals, however, that these arguments are at variance with the proper Maxwell theory and must be regarded as *ad hoc* assumptions carefully chosen so as to achieve the desired result. In this connection it is emphasized that the problem of "acausalities" associated with the Lorentz-Dirac equation are by no means inherent difficulties in classical electron theory, but are procured only through the postulate that this equation represents the *exact* equation of motion for a point electron.

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§ 1. Introduction

In Classical Electron Theory, as based on the pioneering work of ABRAHAM and LORENTZ, the electron is conceived of as a minute spherical distribution of an in principle infinite number of infinitesimal electrified constituents. From the very beginning it was, of course, realized that an explanation of the *stability* of such a system was outside the purview of the Maxwell theory, but nevertheless the hope was entertained that once the stability was taken for granted, a consistent scheme could be developed in which empirically well established phenomena like emission of radiation, the presence of radiation damping and — perhaps — even the inertia of the electron were unambiguous consequences of the mutual interaction of the constituent corpuscles. Thus, fundamentally the classical electron is to be regarded as a system of infinitely many mechanical degrees of freedom (in addition to the degrees of freedom of the field).

Within the framework of a non-relativistic description the reduction of the number of mechanical degrees of freedom to six, characterizing the mechanical phase of a single charged particle, presents no difficulties, amounting merely to the introduction of appropriate assumptions regarding the rigidity of the charge distribution. However, within a proper relativistic scheme this situation is radically different owing to the finite propagation velocity of all physical actions, referred to as "retardation". Indeed, within the framework of the Maxwell theory, any charge distribution — however limited its spatial extension — retains the full complexity associated with a system of infinitely many degrees of freedom*. This circumstance is especially conspicuous in the formulation of the detailed energy-momentum balance, where the energy-momentum tensor, not only of the electromagnetic field but of the "mechanical" part of the system as well, presents the natural tool.

An attempt to formulate a relativistic description of a point electron, characterized exhaustively by the parameters fixing its mechanical phase,

* A similar feature appears in quantum electrodynamics. Here the infinitely many degrees of freedom manifest themselves through excitations of the "electron field".

was initiated by DIRAC[1] in his well-known treatise "Classical Theory of Radiating Electrons", which was followed by several other contributions, notably by FEYNMAN and WHEELER[2] and by ROHRLICH[3]. Basic to these departures from the conventional scheme are the endeavours to express the equation of motion as well as the conservation laws as four-vector relations involving only the momentary velocity, acceleration (and higher derivatives) of the particle besides its phenomenologically introduced mass and the external force. Within such a scheme, however, the phenomenon of damping now poses a problem and can, in fact, only be treated at all on the basis of recipes specially concocted for that very purpose.

Surveying the numerous discussions in the litterature, it is hard to overlook the presence of a certain confusion arising primarily from a lack of sufficient care in distinguishing between the rigorous consequences of the Maxwell theory on the one hand, and the conclusions — e.g. concerning acausalities in the equation of motion — reached by Dirac and his followers by transcending this framework, on the other.

The series of studies here undertaken represents an attempt to arrive at a clearer understanding in this respect. Part I, presented on the following pages, is concerned with the energy-momentum balance in processes where radiative phenomena are important, whereas part II, soon to be published, deals with the "mechanical" or "adiabatic" approximation and analyses from various angles the question of the transformation properties of electromagnetic energy and momentum. Finally, part III is intended to illustrate some of the general principles through the specific example of a charge in hyperbolic motion.

Although the discussion of consistency problems is an essential aspect of the present study, we have not followed the axiomatic approach, preferred by some workers, starting with rigorous definitions of concepts like radiation. Notwithstanding the intrinsic interest of such attempts, they appear to convey an unwarranted impression of freedom in the choice and definition of concepts on which to base our description of nature. Thus, we have preferred to proceed by analysing some idealized examples which are simple enough to allow of a detailed analytical treatment and still sufficiently general to demonstrate typical features in the mechanism of the energy-momentum balance in radiation processes. This attitude towards the consistency problem was greatly influenced by the general lesson of quantum theory, which entailed a serious warning against a priori definitions of physical concepts.

§ 2. Energy-Momentum Balance for Corpuscular Systems in Closed Processes

The concept of a field, possessing independent dynamical degrees of freedom and acting as mediator of interactions between material bodies, springs naturally from the attempt of basing the description of this interaction on customary mechanical ideas, even under circumstances when the retardation of all physical actions plays an essential role. Thus, in electrodynamics, where such a program has met with a far-reaching success, the notion of propagating fields, carrying a well-defined amount of energy, has found a domain of unambiguous applicability, whereas in the General Theory of Relativity, which entails a certain renunciation with respect to the applicability of the usual mechanical concept of force, the attribution of energy to the gravitational field is in general ambiguous. Nevertheless, to the extent that ordinary mechanical ideas may also here serve as a point of departure, the field concept has the same status as in electrodynamics[4)].

For historical reasons, the field concept is often related to the rejection of the idea of forces acting at a distance. This view implies that also static electric or magnetic fields are considered as proper dynamical systems in which a well-defined amount of energy is localized, even though they have, of course, no independent dynamical role in the account of the energy-momentum balance. This situation is essentially different, when time-varying fields are considered, since the question at issue now concerns the possibility of upholding the customary idea of conservation of energy and momentum, rather than a more or less justified prejudice against "action-at-a-distance".

*

As a point of departure, let us recall the familiar account of the energy balance in electrostatic systems. Consider a charge Q, which is divided into a very large number, N, of small charge elements δq_a, situated at the positions $\vec{r}_a$. The total energy of the system, defined as the external work required to build it up adiabatically, is then given by

$$W = \frac{1}{2}\sum_{a \neq b}^{N} \frac{\delta q_a \, \delta q_b}{|\vec{r}_a - \vec{r}_b|}. \tag{1}$$

The feature to be noticed in this expression is the absence of the self-energy of the constituents. In fact, for increasing N, the self-interaction of the constituents decreases relative to their mutual interaction and vanishes in the limit of a continuous charge distribution. Indeed, this feature is merely the formal expression of the observation, already alluded to in the Introduction, that the atomicity of charge is a foreign element in the classical Maxwell theory. Within this framework, properties like electromagnetic self-mass or self-angular-momentum originates in the mutual interactions between the infinitesimal constituents of the system in question, whereas the constituents themselves, according to their very definition, possess no self-energy, self-angular-momentum or similar properties. It is essential to realize that this line of argumentation does not imply any resignation as regards the scope and domain of applicability of the classical Maxwell theory, but rather serves to remind us about the conceptual framework within which any consistent use of the theory must remain.

Rewritten in continuum language the expression (1) takes the familiar form

$$W = \tfrac{1}{2} \int d\vec{x}\, \varrho(\vec{x})\, \varphi(\vec{x}), \tag{2}$$

where $\varrho(\vec{x})$ is the charge density and $\varphi(\vec{x})$ the potential, or expressed in terms of the electric field $\vec{E}(x)$:

$$W = \frac{1}{8\pi} \int d\vec{x}\, \vec{E}^{\,2}(\vec{x}). \tag{3}$$

Although the relation (3), in agreement with Pointings theorem, may be formally interpreted as the integral over an energy density, the derivation provides no basis for conclusions regarding the possibilities for an ascertainment of the presence of energy localized in the field. On the contrary, eq. (3) must so far be regarded merely as a recipe for evaluating the total electromagnetic energy of the system.

As an illustration consider the contrivance recently discussed by Møller[5)], consisting of a small condensor suitably charged so as to cancel a given external electric field within the spatial domain between the plates. This example might convey the impression that, since the electrostatic field energy within the domain considered in this manner can be converted into mechanical energy without noticeable influence on the field outside the condensor, it has indeed been demonstrated that the energy in question was localized in the domain covered by the condensor. However, as must be evident from the beginning, electrodynamics provides no basis for such a conclusion.

Imagine for definiteness a small uncharged condensor which is slowly carried from infinity and placed at a distance R from a charge Q, assuming R to be large compared to the dimensions of the condensor, so that the

Coulomb field from the charge Q within the plates may be approximated by a homogeneous electric field. Under these conditions the displacement of the uncharged condensor from infinity to the distance R does not require external work.

Next, in order to neutralize the Coulomb field inside the condensor, the plates are connected by a conducting wire, and since the two plates coinside with different potential surfaces, a current will flow through the wire until the appropriate charge has been carried from one plate to the other. The current may be utilized to drive a mechanical device, thereby converting the electric field energy into mechanical work.

The charge, q, on the condensor, after the compensation of the field, is given by

$$4\pi\frac{q}{A} = \frac{Q}{R^2}, \tag{4}$$

A denoting the area of the plates, and the total energy gained is thus

$$\Delta U = \frac{1}{8\pi} A d \frac{Q^2}{R^4}, \tag{5}$$

where d signifies the distance between the plates.

Suppose now that it is possible to neglect the field modification outside the condensor. Then, after having cut the conducting wire, one may remove the charge Q to infinity without performing external work, being in the end left with a charged condensor, from which the energy (5) could once more be gained. Thus, it is clearly necessary to take into account that after the original charging up of the condensor, the outside field is modified. In fact, the condensor behaves as a small dipole bound in the Coulomb field from the charge Q. The binding energy $\Delta\varphi$ can be estimated as

$$-\Delta\varphi = -\frac{qQ}{R} + \frac{qQ}{R+d} \simeq -qQ\frac{d}{R^2}, \tag{6}$$

or by means of eqs. (4) and(5)

$$\Delta\varphi = 2\Delta U. \tag{7}$$

Instead of cutting the conducting wire before the removal of the charge Q, the connection between the plates could have been maintained, whereby the field between the plates would have been cancelled at each instance, implying the relation (4) to hold for every r during this process. When the charge has been removed to infinity, the condensor is discharged and the total energy gained is now given by eq. (5). Correspondingly, the force on Q now varies as

$$K(r) = \frac{qQ}{r^2} - \frac{qQ}{(r+d)^2} \simeq \frac{2Q^2}{r^5}\frac{Ad}{4\pi}, \tag{8}$$

since

$$q = \frac{A}{4\pi}\frac{Q^2}{r^2}.$$

Thus, as expected, the total mechanical work amounts to:

$$W = \int_R^\infty K(r)\,dr = \frac{1}{8\pi}\frac{Q^2}{R^4}Ad. \tag{9}$$

This example illustrates how the energy balance for electrostatic systems can be exhaustively accounted for in terms of the customary mechanical concept of potential energy without any reference to the field. Quite generally, within the framework of electrodynamics the impartation of energy to a static field is purely conventional in so far as the energy in question may alternatively be expressed in terms of the co-ordinates of the charged particles.

* *

Quite a different situation is met with in the case of time-varying charge and current distributions. Due to the retardation of physical actions, the field now represents independent degrees of freedom of the total system, which can only be ignored or eliminated at the expense of giving up the notion of energy-momentum balance. As a simple illustration* consider two particles of charge Q—originally at rest at a relative distance $2r_i$—, which are moved simultaneously and symmetrically towards each other** to a relative distance $2r_f$ ($r_f < r_i$), where they stay at rest (see figure 1). If the process is carried out adiabatically, the external work performed equals the change in potential energy

$$W_{ad} = \frac{Q^2}{2r_f} - \frac{Q^2}{2r_i}. \tag{10}$$

If, however, the process is carried out in a finite time, the work required will, as a consequence of the retardation, differ from W_{ad}.

Suppose that the duration of the process, Δt, is chosen so that

$$r_i - r_f < c\Delta t \leq r_i + r_f, \tag{11}$$

* The following example was already discussed in reference 4. Since, however, it shall be utilized here for other purposes, it is reproduced for the convenience of the reader.

** Since the entire discussion is carried out within the framework of special relativity, the freedom to invoke the actions of arbitrary external forces is exploited throughout.

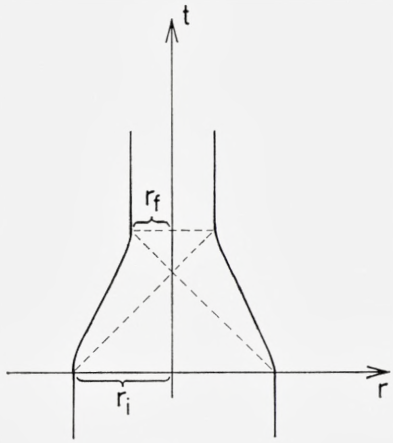

Fig. 1.

which implies that the electromagnetic force on each particle due to the other one during the entire motion is given by the original static Coulomb field. In this case the work required to overcome the electrostatic repulsion only amounts to*

$$2\left(\frac{Q^2}{r_i + r_f} - \frac{Q^2}{2r_i}\right). \tag{12}$$

The very fact that this work differs from the change in potential energy (10) faces us with the choice of either giving up the customary idea of energy conservation, or recognizing the existence of some non-conservative force acting on each particle, independently of the motion of the other since during the process considered no communication is possible between the particles. Within the customary mechanical framework the non-conservative character of this so-called "damping force" is interpreted as a manifestation of an independent set of degrees of freedom, with which the particles may interact and exchange energy, the damping force being just a phenomenological way of taking this interaction into account.

Reconsidering now the above process in this extended framework, we notice that the external work, W_D, required to overcome the damping force on each particle during the displacement must, for symmetry reasons, be the same for both particles and, according to its definition, independent of the motion of the other. Thus the total energy to be supplied is not given by eq. (12) but by the relation

* If the retarded interaction were replaced by a time-symmetric interaction, clearly, the work performed would also in this case equal W_{ad}.

$$W = 2\left(\frac{Q^2}{r_i + r_f} - \frac{Q^2}{2r_i} + W_D\right), \tag{13}$$

where W_D is related to the hypothetical "radiation energy" $\mathscr{E}_R$ by the requirement of energy balance

$$\mathscr{E}_R + \frac{Q^2}{2r_f} = W + \frac{Q^2}{2r_i}. \tag{14}$$

Hence:

$$2W_D - \mathscr{E}_R = Q^2 \frac{(r_i - r_f)^2}{2r_i r_f (r_i + r_f)}. \tag{15}$$

Whereas this expression is still compatible with a complete absence of radiation, corresponding to $\mathscr{E}_R = 0$, evidently, $\mathscr{E}_R$ and W_D cannot *both* vanish. Furthermore, the fact that, according to the initial conditions, $\mathscr{E}_R \geqslant 0$, implies that W_D is positive definite, reflecting the irreversible character of the process of radiation emission.

Face now the particular case in which the equality sign in eq. (11) holds, i.e.

$$c\Delta t = r_i + r_f \tag{16}$$

and introduce the average velocity v and average acceleration g through the relations

$$\left.\begin{aligned} v &= \frac{\Delta r}{\Delta t} = c\frac{r_i - r_f}{r_i + r_f} \\ g &= \frac{\Delta r}{(\Delta t)^2} = c^2\frac{r_i - r_f}{(r_i + r_f)^2}. \end{aligned}\right\} \tag{17}$$

Then eq. (15) may be rewritten in the suggestive form

$$2W_D - \mathscr{E}_R = 2\frac{Q^2}{c^3}\frac{g^2}{1 - v^2/c^2}\Delta t. \tag{18}$$

So far no conclusions as to the individual value of W_D and $\mathscr{E}_R$ can be drawn. However, since W_D, as already noticed, is independent of the motion of the other particle, it may be determined by considering another process, in which only one of the particles is displaced along the same world line as before, whereas the other is kept fixed. Denoting by E_R the energy transferred to the radiation field during this process, the energy balance now yields the relation:

$$E_R + \frac{Q^2}{r_i + r_f} = \tfrac{1}{2} W + \frac{Q^2}{2 r_i}, \tag{19}$$

where W is given by eq. (13) as before. Hence it follows that

$$E_R = W_D. \tag{20}$$

Since the role of the fixed charge in this process is purely auxiliary, we may conclude that whenever a charge, Q, is displaced a distance Δr during a time Δt, being at rest outside this time interval, a net external work equal to W_D has to be performed*.

Furthermore, it follows from eq. (18) that**

$$E_R = W_D \geqslant \frac{Q^2}{c^3} \frac{g^2}{1 - v^2/c^2} \Delta t. \tag{21}$$

The above example is well suited to exhibit the futility of attempts to achieve a pictorial representation of the energy transfer to the radiation field as an emission process localized in space and time***. In fact, such a picture would entail that since a single charge during the displacement (the other being kept fixed) emits the amount of energy $E_R = W_D$, — which, as far as the particle degrees of freedom are concerned, is irreversibly lost once the particle comes to rest again — it would emit the same amount of energy, even if the second particle were simultaneously displaced, since the first particle could only recognize this displacement after the completion of its own act of emission. Thus, the conclusion would be — in conflict with the result (15) — that the total amount of energy emitted equaled $2 W_D$, and furthermore, provided the original distance $2 r_i$ is chosen larger than $2c\,\Delta t$, that this energy, immediately after the particles had come to rest, were localized within two non-overlapping spherical shells.

Thus a proper account of the energy balance in the process considered cannot be given within the picture indicated but must start from the recognition that a radiation process, far from having the character of a localized event, manifests a modification of the field as a whole. Although at the

* Since only the Coulomb potential enters into the above example, it may appear puzzling how to carry through the argumentation in the Coulomb Gauge. The solution to this conundrum was confided to us by JENS LINDHARD.

** In the non-relativistic limit minimization of the integral $(2Q^2/3c^3) \int_0^{\Delta t} \ddot{x}^2\, dt$, subject to the appropriate boundary conditions, actually gives 8 times the value (21). However, considering more cunning contrivances involving several charges, it is easy to increase the lower bound of the inequality (21).

*** For a detailed exposition of such attempts, see ref. 3 & 6.

termination of the displacement of a particular charge, a definite amount of work has been performed on this charge independently of a possible displacement of the other, still the fraction of this supplied energy, ultimately to appear in the radiation field, is at this moment to a certain extent indeterminate, being dependent on whether or not the second particle is actually displaced.

* * *

The preceding analysis has demonstrated how the combined requirements of energy conservation and retardation entails the existence of the phenomenon of damping for a single finite charge. Thus, a detailed account of the energy balance associated with the mutual interaction between the infinitesimal constituents of the charge, each of which suffers no damping, must necessarily substantiate this conclusion. Furthermore, the analysis suggests that the total damping acting on the charge may be pictorially represented as the accumulated effect of the continual lack of "adjustment" of action and reaction in the mutual forces between the corpuscles, brought about by the impossibility of instantaneous communication between them.

To trace the problem further back is clearly impossible within the framework of the Maxwell theory, since it would amount to deducing the retarded, as opposed to the advanced, character of the electromagnetic interaction, a program, the very formulation of which would involve a contradiction in terms within a scheme which implicitly assumes the freedom to influence the behaviour of the charges or charge elements in question*.

With the purpose of obtaining a quantitative expression** for the damping, let us evaluate the total four-momentum to be supplied to constrain a system of electrified corpusles with a given total charge, to perform a cyclic process, where in the initial and final state the corpuscles are at rest in a given configuration. Since an adiabatic process is of no consequence for the problem at issue, we may for simplicity assume that this configuration is originally built up by adiabatically assembling the corpuscles from rest at infinity, and finally adiabatically decomposed by removing the corpuscles back to infinity.

The fact that a finite amount of energy-momentum has to be supplied at all in a cyclic process clearly reflects the limitations in the usual form of the law of action and reaction in situations where the retardation must be

* See in this connection the further remarks on page 32 ff.

** In the following x denotes the four-vector $(\vec{x}, it)$ and a similar notation is employed for other four-vectors, the scale of length being chosen as the distance travelled by a light signal per unit of time. Also, for the sake of clarity all tensor indices have been suppressed, since it will be clear from the context whether matrix multiplication or scalar products are implied. Where ambiguities may occur, scalar products are indicated by a dot.

taken explicitly into account. Indeed, the total four-momentum $P_R^{(1,2)}$ which a corpuscle 2 via its retarded field communicates to another corpuscle 1 during the process considered does *not* equal minus the four-momentum $P_R^{(2,1)}$ which the first corpuscle via its retarded field communicates to the second. Instead, the generalized law of action and reaction may for a closed, cyclic process be expressed as

$$P_R^{(1,2)} + P_A^{(2,1)} = 0 \tag{22}$$

where $P_A^{(2,1)}$ signifies the four-momentum communicated to the second corpuscle via the *advanced* field of the first.

To demonstrate this symmetry, consider a single pair of corpuscles. The four-momentum $P_R^{(1,2)}$ communicated during the process to the first corpuscle via the retarded field of the second is given by

$$P_R^{(1,2)} = \oint dx \mathscr{F}_R^{(2)}(x) s_1(x), \tag{23}$$

where $\mathscr{F}_R^{(2)}(x)$ denotes the field tensor corresponding to the retarded electromagnetic field generated by the second corpuscle, and

$$s_1(x) = \delta q_1 \int d\tau_1 \delta^{(4)}(x - x_1(\tau_1)) U_1(\tau_1) \tag{24}$$

the charge current density associated with the motion of corpuscle one along its world line $x_1(\tau_1)$; $U_1 = \dfrac{dx_1}{d\tau_1}$. Similarly

$$P_A^{(2,1)}(x) = \oint dx \mathscr{F}_A^{(1)}(x) s_2(x), \tag{25}$$

where $\mathscr{F}_A^{(1)}(x)$ denotes the advanced electromagnetic field generated by the first corpuscle.

Expressing $\mathscr{F}_R^{(2)}$ and $\mathscr{F}_A^{(1)}$ in terms of the currents $s_2(x)$ and $s_1(x)$ by means of the retarded and advanced Green's functions $\mathscr{D}_R(x)$ and $\mathscr{D}_A(x)$, and remembering that $\mathscr{D}_R(x) = \mathscr{D}_A(-x)$, one obtains*:

$$\left.\begin{aligned} P_R^{(1,2)} + P_A^{(2,1)} &= 4\pi \oint dxdy \{[\partial_x \mathscr{D}_R(x-y) \wedge s_2(y)]\, s_1(x) + [\partial_y \mathscr{D}_A(y-x) \wedge s_1(x)] s_2(y)\} \\ &= 4\pi \oint dxdy \{[\partial_x \mathscr{D}_R(x-y) \wedge s_2(y)] s_1(x) + [s_1(x) \wedge \partial_x \mathscr{D}_R(x-y)] s_2(y)\} \\ &= -4\pi \oint dxdy\, [s_2(y) \wedge s_1(x)]\, \partial_x \mathscr{D}_R(x-y), \end{aligned}\right\} \tag{26}$$

where the last equality follows from the identity

* For the antisymmetric tensor $a_i b_j - a_j b_i$ constructed from two four-vectors a and b, we employ the customary notation $a \wedge b$.

$$[a \wedge b]\, c + [c \wedge a]\, b + [b \wedge c]\, a = 0 \tag{27}$$

valid for any three four-vectors, a, b, c. Finally, taking advantage of the continuity equation for the currents, the expression (26) is immediately transformed to a surface integral, which vanishes by virtue of the boundary conditions*

$$\left. \begin{aligned} & P_R^{(1,2)} + P_A^{(2,1)} = \\ & \quad - 4\pi \oint dx dy \left\{ s_2(y) \partial_x \cdot (s_1(x) \mathscr{D}_R(x-y)) + s_1(x) \partial_y \cdot (s_2(y) \mathscr{D}_R(x-y)) \right\} = 0. \end{aligned} \right\} \tag{28}$$

It may be remarked in passing that this result may also be immediately obtained by variation of the translational invariant quantity

$$\iint\limits_{-\infty}^{+\infty} d\tau_1 d\tau_2 \mathscr{D}_R(x_1(\tau_1) - x_2(\tau_2)) U_1(\tau_1) \cdot U_2(\tau_2). \tag{29}$$

Continuing the evaluation of the damping, it is next noticed that, since according to their very definition, the corpuscles suffer no damping, the equations of motion for two constituents are

$$\left. \begin{aligned} \delta m_1 g_1 &= F_1^{(\text{el})} + K_1^{(\text{ext})} \\ \delta m_2 g_2 &= F_2^{(\text{el})} + K_2^{(\text{ext})}, \end{aligned} \right\} \tag{30}$$

where $F_1^{(\text{el})}$ ($F_2^{(\text{el})}$) denotes the electromagnetic four-force generated by the second (first) corpuscle, whereas $K_1^{(\text{ext})}$ ($K_2^{(\text{ext})}$) stands for the external force required to constrain the corpuscles in question to perform the prescribed cyclic motion. Hence the total four-momentum P supplied during the process, amounts to

$$\left. \begin{aligned} P &= \oint \left\{ K_1^{(\text{ext})} d\tau_1 + K_2^{(\text{ext})} d\tau_2 \right\} \\ &= - \oint \left\{ F_1^{(\text{el})} d\tau_1 + F_2^{(\text{el})} d\tau_2 \right\} \\ &= - \left(P_R^{(1,2)} + P_R^{(2,1)} \right) \end{aligned} \right\} \tag{31}$$

where $P_R^{(1,2)}$ ($P_R^{(2,1)}$) is given by eq. (23). Now utilizing the generalized relationship (22) between action and reaction, the expression for P may be rewritten as

$$\left. \begin{aligned} P &= - P_R^{(1,2)} + P_A^{(1,2)} \\ &= - \oint dx [\mathscr{F}_R^{(2)}(x) - \mathscr{F}_A^{(2)}(x)] s_1(x). \end{aligned} \right\} \tag{32}$$

It should be emphasized that neither this result, nor the relation (22) on which it is based, is valid differentially, but only holds for the entire cyclic process.

* A similar result is obtained by Feynman and Wheeler (loc. cit), who, however, attempt to interpret the relation differentially, in stead of maintaining the integral form.

Finally, summing the contributions from the infinitely many charged constituents of the system, and remembering that the importance of the "self-terms" is negligible compared to the interaction terms, one obtains for the total expenditure of four-momentum during the process

$$\left.\begin{aligned}\mathscr{P} &= -\tfrac{1}{2}\sum_{a \neq b}\oint dx[\mathscr{F}_R^{(b)}(x) - \mathscr{F}_A^{(b)}(x)]s_a(x)\\ &= -\tfrac{1}{2}\oint dx[\mathscr{F}_R(x) - \mathscr{F}_A(x)]s(x),\end{aligned}\right\} \quad (33)$$

where $s(x)$ now signifies the total four-current density, and $\mathscr{F}_{R,A}(x)$ the total electromagnetic field.

This result, which is valid for an arbitrary finite charge distribution, represents a rigorous consequence of the conventional Maxwell theory, and the appearance in the integrand of the advanced fields raises no problem of interpretation. Indeed, it is clear that the derivation provides no basis for attributing to the difference $\mathscr{F}_R(x) - \mathscr{F}_A(x)$ the status of a measurable field; the occurrence of this difference is merely dependent on the formal artifice of exploiting the symmetry between the retarded and advanced Green's functions so as to combine in a special way the contributions in eq. (31) from the mutual interactions to the net energy-momentum expenditure.

The general expression (33) may in particular be applied to the case of a charge distribution so limited in spatial extension, that the difference $\mathscr{F}_R(x) - \mathscr{F}_A(x)$ (which is regular even in the point limit on the world line of the source) may be expanded in terms of the dimensions of the system. The well-known result*, first derived by Dirac, for the difference $\mathscr{F}_R^{(a)}(x) - \mathscr{F}_A^{(a)}(x)$, in the immediate vicinity of the world-line of the a'th corpuscle, is

$$\mathscr{F}_R^{(a)}(x) - \mathscr{F}_A^{(a)}(x) \simeq \tfrac{4}{3}\delta q_a[U_a \wedge \dot{g}_a], \quad (34)$$

where the terms neglected vanish as the point x approaches the world-line. Inserting this relation into equation (33), and using the expression (24) for the current density, one finds for the total four-momentum expenditure in the limit when all the world-lines of the corpuscles become identical

$$\mathscr{P} = \tfrac{2}{3}Q^2\oint\{g^2U - \dot{g}\}d\tau \quad (35)$$

(the second term in the integrand does of course not contribute in a cyclic process).

* * * *

* For completeness a slightly simplified version of Dirac's derivation is given in appendix A.

Since the previous considerations were solely concerned with cyclic processes, the entire energy-momentum expenditure was of course of irreversible character. For the following discussion it is essential to generalize the considerations so as to embrace instances in which the energy and momentum of the system also suffer a reversible change. Thus, consider a process, in which a pair of corpuscles is brought from rest at infinite separation along arbitrary world-lines to a state of common uniform motion. From the equations of motion (30) the total energy and momentum to be supplied is now given by

$$\left.\begin{aligned} \Delta P(T) &= \int_{-\infty}^{T} K_1^{(\mathrm{ext})}\, d\tau_1 + \int_{-\infty}^{T} K_2^{(\mathrm{ext})}\, d\tau_2 \\ &= (\delta m_1 + \delta m_2)\Delta U - \int_{-\infty}^{T} dx \{\mathscr{F}_R^{(2)}(x) s_1(x) + \mathscr{F}_R^{(1)}(x) s_2(x)\}, \end{aligned}\right\} \quad (36)$$

where ΔU denotes the change in the four-velocity. The integral is evaluated in appendix B, and the ensuing result for $P = P_{\mathrm{initial}} + \Delta P$ is

$$\left.\begin{aligned} P(T) &= (\delta m_1 + \delta m_2) U \\ &+ \frac{\delta q_1 \delta q_2}{\varepsilon} U + \frac{\delta q_1 \delta q_2}{\varepsilon^3}\left(\frac{l_4}{i\gamma}\right) l - \int_{-\infty}^{t_{12}} dx [\mathscr{F}_R^{(2)}(x) - \mathscr{F}_A^{(2)}(x)] s_1(x), \end{aligned}\right\} \quad (37)$$

where it is understood that the final state of common uniform motion has been reached at least a time $2\gamma\varepsilon$ prior to T, ε being the rest-distance between the corpuscles. Furthermore l denotes the four-vector of length ε joining the two world lines, perpendicularly to the common four-velocity U, and t_{12} is defined such that a light signal emitted from the first corpuscle at that time will reach the second corpuscle at time T (see figure 2). Finally, $i\gamma = U_4$.

Provided the system becomes isolated at time T in the laboratory by simultaneous removal of the external constraining forces, the energy and momentum of the system is given by the expression (37). Clearly the non-covariant appearance of the third term in this formula is associated with the fact that the plane T = constant is not intrinsically related to the world lines. In the present case of common uniform motion, however, the plane perpendicular to the four-velocity is evidently singled out, and it can therefore be expected, that if in the integration leading to equation (37) the plane T = constant, is replaced by the plane $\mathring{T}$ = constant (corresponding to removal of the external forces simultaneously in the rest-frame) the resulting expression $P_{(\mathring{T})}$ would "appear more covariant". Indeed, it is evident (see figure 2) that

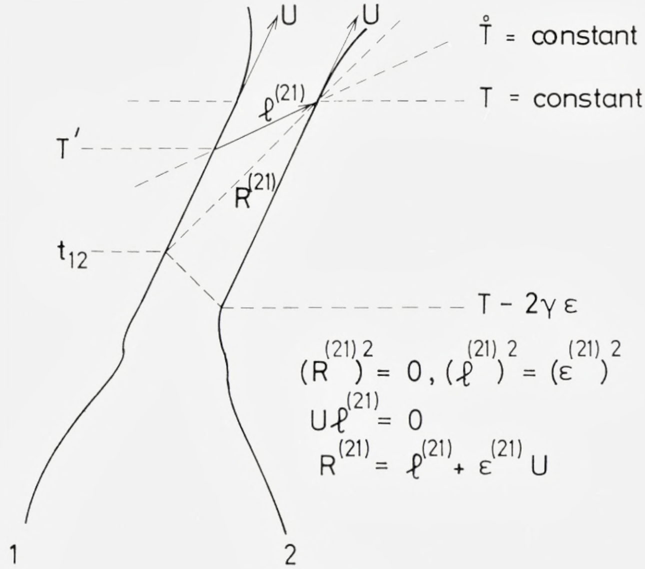

Fig. 2.

$$\left.\begin{aligned} \Delta P_{(\mathring{T})} &= \int_{-\infty}^{T'} K_1^{(\mathrm{ext})}\, d\tau_1 + \int_{-\infty}^{T} K_2^{(\mathrm{ext})}\, d\tau_2 \\ &= \Delta P(T) + \delta q_1 \int_{T'}^{T} \mathscr{F}_R^{(2)}(x_1(\tau))\, U_1(\tau)\, d\tau \\ &= \Delta P(T) + \delta q_1 \delta q_2 \left(\frac{-l_4}{i\gamma}\right) \frac{[U \wedge l]}{\varepsilon^3}\, U, \end{aligned}\right\} \quad (38)$$

where the last step is justified since the motion in the time interval concerned is uniform. Hence, $P_{(\mathring{T})} = P_{\mathrm{initial}} + \Delta P_{(\mathring{T})}$ is given by

$$P_{(T)} = (\delta m_1 + \delta m_2)\, U_{(\mathring{T})} + \frac{\delta q_1\, \delta q_2}{\varepsilon}\, U_{(\mathring{T})} - \int_{-\infty}^{t_{12}} dx\, [\mathscr{F}_R^{(2)}(x) - \mathscr{F}_A^{(2)}(x)]\, s_1(x). \quad (39)$$

It is important to realize that equation (39) should not be construed as a re-definition of the energy and momentum of the original system, but that it represents the true energy and momentum of a *different* system, namely one prepared by removing the constraining agents at different instances T' and T (as judged from the laboratory system) for the two corpuscles. Thus, the symbol $U_{(\mathring{T})}$ is meant to indicate that the four-velocity U is achieved by the two corpuscles at times T' and T, respectively.

Extending the above considerations to the case of a large number of corpuscles of identical charge, uniformly distributed on an infinitesimal spherical shell of radius ε in the common rest-frame, one obtains in the continuum limit in place of eq. (37):

$$\left.\begin{aligned} \mathscr{P}(T) &= \tfrac{1}{2}\sum_{a\neq b}\Big\{(\delta m_a+\delta m_b)U+\frac{\delta q_a\,\delta q_b}{\varepsilon_{ab}}U \\ &+\frac{\delta q_a\,\delta q_b}{\varepsilon_{ab}^3}\left(\frac{l_4^{(ab)}}{i\gamma}\right)l^{(ab)}-\int_{-\infty}^{t_{ab}}dx[\mathscr{F}_R^{(b)}(x)-\mathscr{F}_A^{(b)}(x)]s_a(x)\Big\} \\ &= MU+\frac{Q^2}{2\,\varepsilon}\gamma\left(\tfrac{4}{3}\vec{v},\, i(1+\tfrac{1}{3}v^2)\right)+\tfrac{2}{3}Q^2\int_{-\infty}^{\tau(T)}g^2\,U d\tau, \end{aligned}\right\} \quad (40)$$

where we have used the relation (34) and where $M=\sum\delta m$. Similarly eq. (39) is replaced by

$$\mathscr{P}_{(\mathring{T})} = MU_{(\mathring{T})}+\frac{Q^2}{2\,\varepsilon}U_{(\mathring{T})}+\tfrac{2}{3}Q^2\int_{-\infty}^{\tau}g^2\,U\,d\tau. \quad (41)$$

Defining the electromagnetic energy-momentum tensor for the system of electrified corpuscles

$$\mathscr{T}_{ik} = \sum_{a\neq b}\frac{1}{4\pi}\{\mathscr{F}_{im}^{(a)}\mathscr{F}_{mk}^{(b)}-\tfrac{1}{4}\delta_{ik}(\mathscr{F}^{(a)}\mathscr{F}^{(b)})\} \quad (42)$$

(where of course in the continuum limit the restriction on the summation may be dropped), the four-momentum $\mathscr{P}(T)$ may alternatively be expressed as*

$$\mathscr{P}(T) = MU-\sum_{a\neq b}\int^{T}dx\,\mathscr{F}_R^{(b)}(x)\,s_a(x) = MU-\int^{T}dx\;\partial_x\cdot\mathscr{T}(x) \quad (43)$$

or

$$\mathscr{P}(T) = MU+\int_{T=\text{const.}}\mathscr{T}\cdot d\Gamma, \quad (44)$$

where $d\Gamma_4 = id\vec{x}$.

Similarly, the expression (41) for the four-momentum $\mathscr{P}_{(\mathring{T})}$ may be written

$$\mathscr{P}_{(\mathring{T})} = MU_{(\mathring{T})}+\int_{\mathring{T}=\text{const.}}\mathscr{T}\cdot d\Omega. \quad (45)$$

* Actually, the ensuing expressions (44) and (45) are slightly more general than the corresponding expression (40) and (41), since the former do not presuppose the corpuscles to be in uniform motion at the moment considered.

Of course, once the system, prepared according to the prescription (45), has become isolated, it is possible to replace the plane $\mathring{T}$ = constant with the plane T = constant without changing the value of the constant four-vector $\mathscr{P}_{(\mathring{T})}$ However, on the plane T = constant the corpuscles would in general *not* move with the common velocity $U_{(\mathring{T})}$ and the value of the electromagnetic energy-momentum tensor would have changed accordingly. Thus, it is important to bear in mind that even in the point limit, when the charge is concentrated within a vanishingly small spatial domain, the two quantities $\mathscr{P}(T)$ and $\mathscr{P}_{(\mathring{T})}$ remain quite distinct in value as well as in physical content*.

§ 3. Differential Energy-Momentum Balance and Equations of Motion for a Point Charge

In accordance with the plan outlined in the Introduction, we now proceed to the contraposition of the conclusions arrived at in the previous paragraph, based on the conventional Maxwell theory, with the endeavours initiated by Dirac[1)] to develop a classical description of an ideal point electron. Although the inference drawn in the earlier discussion regarding the occurrence of damping acting on the individual charge remains valid, the question as to the origin of this phenomenon now requires a new answer. Indeed, since the mentioned inference did only depend on the principles of retardation and energy conservation, it is clear that — unless at least one of these general principles is abandoned** — the introduction of the notion of an ideal point charge immediately creates the need for the concoction of a recipe to account for the damping, a direct analysis being excluded by the very quality of this notion. Needless to say, there is a considerable freedom in selecting the way in which the classical electron theory is adapted to this new concept.

To exhibit most clearly the essential differences between the theory proposed by Dirac and the conventional description, it is advantageous not to follow directly the path trodden by him, but to proceed in a manner which at every step permits of a comparison between the two different schemes. Hence, the plan of action for the ensuing section is as follows:

Imagine that in each of the momentary rest systems corresponding to the motion of the electron, the point charge is surrounded by a sphere of

* A thorough discussion of this aspect of the problem will be found in Studies in Classical Electron Theory II.

** Cf. the later remarks on the work by Feynman & Wheeler[2)].

vanishingly small radius ε, and evaluate in the momentary rest-frame the total electromagnetic momentum and energy $(\overset{\circ}{\vec{\mathscr{P}}}_{\Omega}, \overset{\circ}{\mathscr{E}}_{\Omega})$ formally associated with the region, Ω, exterior to this tiny sphere. Follow next Dirac and his school in — explicitly or implicitly — exploiting the freedom gained through the introduction of the notion of a "point charge" to re-define the electromagnetic energy and momentum of the system so as to transform as a four-vector, namely that four-vector whose components in the momentary rest-frame are given by $(\overset{\circ}{\vec{\mathscr{P}}}_{\Omega}, i\overset{\circ}{\mathscr{E}}_{\Omega})$ just introduced. Finally, contemplate for comparison the similar problem, within the conventional scheme, of evaluating the electromagnetic momentum and energy $(\vec{\mathscr{P}}_{\Gamma}, \mathscr{E}_{\Gamma})$ associated with the domain Γ exterior to the Heaviside ellipsoid corresponding to a sphere of radius ε in the momentary rest-frame. Of course, the quantity $(\vec{\mathscr{P}}_{\Gamma}, i\mathscr{E}_{\Gamma})$ is not a four-vector, as is evident from the fact that the electromagnetic energy-momentum tensor is not separately divergence-free.

*

To carry into effect this plan, consider the motion of a charged particle interacting with an incoming external source-free electromagnetic field, $\mathscr{F}_{(in)}$ and focus the attention on that part of the total electromagnetic field, which is causally connected to a definite segment of the particle trajectory, corresponding to two successive positions of the particle $\vec{x}(t_1)$ and $\vec{x}(t_2)$ at times t_1 and t_2. Remarkably enough, it is possible to evaluate explicitly the total momentum and energy formally associated with the mentioned part of the field, which is of course confined to the region Ξ (see fig. 3) between two consecutive light spheres centred at the two points $\vec{x}(t_1)$ and $\vec{x}(t_2)$ respectively. Although the momentum and energy of this part of the field has no direct physical significance, the formal expressions obtained are of some interest in themselves and will provide a useful intermediate step for the evaluation of the corresponding quantities associated with the domains of proper interest, Ω and Γ.

The total field may be written as

$$\left.\begin{aligned} \vec{E}(t) &= \vec{E}_R(t) + \vec{E}_{in}(t) \\ \vec{H}(t) &= \vec{H}_R(t) + \vec{H}_{in}(t), \end{aligned}\right\} \quad (46)$$

where the retarded Liénard-Wichert fields $\vec{E}_R$ and $\vec{H}_R$ are given by the familiar expressions

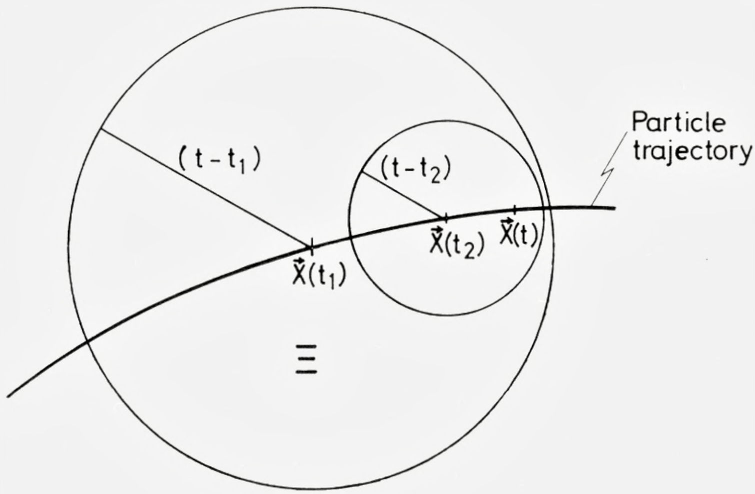

Fig. 3.

$$
\left.\begin{aligned}
\vec{E}_R(t) &= \vec{E}_{\mathrm{I}}(t) + \vec{E}_{\mathrm{II}}(t) \\
&= \frac{Q}{\gamma^2(1-\vec{v}\cdot\hat{n})^3}\frac{n-\vec{v}}{(t-t_R)^2} + \frac{Q}{(1-\vec{v}\cdot\hat{n})^3}\frac{\hat{n}\times[(\hat{n}-\vec{v})\times\vec{g}]}{(t-t_R)} \\
\vec{H}_R(t) &= \vec{H}_{\mathrm{I}} + \vec{H}_{\mathrm{II}}(t) = \hat{n}\times\vec{E}_{\mathrm{I}} + \hat{n}\times\vec{E}_{\mathrm{II}},
\end{aligned}\right\} \quad (47)
$$

the abbreviations $\vec{E}_{\mathrm{I}}$ and $\vec{E}_{\mathrm{II}}$ referring to the first and second term of $\vec{E}_R$ respectively and correspondingly for $\vec{H}_R$. Furthermore $\hat{n}$ denotes the unit vector from the retarded position to the field point and the velocity $\vec{v}$ as well as the acceleration g are to be evaluated at the retarded time t_R; as usual $\gamma^{-2} = 1 - v^2$. The straightforward but cumbersome evaluation of the integrals

$$
\left.\begin{aligned}
\mathscr{E}_{\Xi} &= \frac{1}{8\pi}\int_{\Xi}\{(\vec{E}_{\mathrm{I}}+\vec{E}_{\mathrm{II}}+\vec{E}_{in})^2 + (\vec{H}_{\mathrm{I}}+\vec{H}_{\mathrm{II}}+\vec{H}_{in})^2\}dV \\
\vec{\mathscr{P}}_{\Xi} &= \frac{1}{4\pi}\int_{\Xi}\{(\vec{E}_{\mathrm{I}}+\vec{E}_{\mathrm{II}}+\vec{E}_{in})\times(\vec{H}_{\mathrm{I}}+\vec{H}_{\mathrm{II}}+\vec{H}_{in})\}dV
\end{aligned}\right\} \quad (48)
$$

is deferred to appendix C, but for the purpose of reference the results are quoted here term by term:*

* It is noteworthy that the expressions (49a) and (50a) depend only on the instantaneous velocity of the particle, in spite of the fact that the integrands depend on the entire preceding trajectory.

$$\left.\begin{aligned}
&\left.\begin{aligned}
&\frac{1}{8\pi}\int_{\Xi}\{\vec{E}_{\mathrm{I}}^2+2\vec{E}_{\mathrm{I}}\vec{E}_{\mathrm{II}}+\vec{H}_{\mathrm{I}}^2+2\vec{H}_{\mathrm{I}}\vec{H}_{\mathrm{II}}\}\,dV\\
&\quad=\frac{Q^2}{3}\left[\frac{4\gamma^2(t_2)-1}{2(t-t_2)}-\frac{4\gamma^2(t_1)-1}{2(t-t_1)}\right]
\end{aligned}\right\}\quad\text{(a)}\\
&\frac{1}{8\pi}\int_{\Xi}\{\vec{E}_{\mathrm{II}}^2+\vec{H}_{\mathrm{II}}^2\}\,dV=\tfrac{2}{3}Q^2\int_{t_1}^{t_2}\frac{\vec{g}^2-(\vec{g}\times\vec{v})^2}{(1-v^2)^3}\,dt\quad\text{(b)}\\
&\frac{1}{8\pi}\int_{\substack{\text{all}\\\text{space}}}\{\vec{E}_{in}^2+\vec{H}_{in}^2\}\,dV\equiv\mathscr{E}_{in}=\text{constant}\quad\text{(c)}\\
&\frac{1}{8\pi}\int_{\substack{\text{all}\\\text{space}}}\{2\vec{E}_R\vec{E}_{in}+2\vec{H}_R\vec{H}_{in}\}\,dV=-\int_{-\infty}^{t}\vec{E}_{in}\vec{v}\,dt.\quad\text{(d)}
\end{aligned}\right\}\quad(49)$$

Similarly

$$\left.\begin{aligned}
&\left.\begin{aligned}
&\frac{1}{4\pi}\int_{\Xi}\{\vec{E}_{\mathrm{I}}\times\vec{H}_{\mathrm{I}}+\vec{E}_{\mathrm{I}}\times\vec{H}_{\mathrm{II}}+\vec{E}_{\mathrm{II}}\times\vec{H}_{\mathrm{I}}\}\,dV\\
&\quad=\tfrac{2}{3}Q^2\left[\frac{\gamma^2(t_2)}{t-t_2}\vec{v}(t_2)-\frac{\gamma^2(t_1)}{t-t_1}\vec{v}(t_1)\right]
\end{aligned}\right\}\quad\text{(a)}\\
&\frac{1}{4\pi}\int_{\Xi}\{\vec{E}_{\mathrm{II}}\times\vec{H}_{\mathrm{II}}\}\,dV=\tfrac{2}{3}Q^2\int_{t_1}^{t_2}\frac{\vec{g}^2-(\vec{g}\times\vec{v})^2}{(1-v^2)^3}\vec{v}\,dt\quad\text{(b)}\\
&\frac{1}{4\pi}\int_{\substack{\text{all}\\\text{space}}}\{\vec{E}_{in}\times\vec{H}_{in}\}\,dV=\vec{\mathscr{P}}_{in}=\text{constant.}\quad\text{(c)}\\
&\frac{1}{4\pi}\int_{\substack{\text{all}\\\text{space}}}\{\vec{E}_R\times\vec{H}_{in}+\vec{E}_{in}\times\vec{H}_R\}\,dV=-Q\int_{-\infty}^{t}\{\vec{E}_{in}+\vec{v}\times\vec{H}_{in}\}\,dt.\quad\text{(d)}
\end{aligned}\right\}\quad(50)$$

The constancy of $\mathscr{E}_{in}$ and $\vec{\mathscr{P}}_{in}$ simply expresses that the incoming field $\vec{E}_{in}$, $\vec{H}_{in}$, according to its definition, at all times develops as a free field, independent of the presence of the charge. Thus the fact that the incoming field may nevertheless transfer energy and momentum to the particle, is reflected through the occurrence of interference between this field and the retarded field generated by the charge (cf. equations (49d) and (50d)).

Although the structure of the right-hand side of equations (49b) and (50b), in view of the expression (35), may invite an interpretation of $\vec{E}_{II}$ and $\vec{H}_{II}$ as "the radiation field" carrying at any time the "total radiated four-momentum", it must be clearly recognized that such an interpretation is purely *formal*, already because the quantities $\vec{E}_{II}$ and $\vec{H}_{II}$ do not by themselves satisfy the Maxwell equations. As far as equations (49a) and (50a) are concerned, the curious form of the right-hand sides is clearly connected to the fact, that the boundaries of the region Ξ are spheres whose centres are displaced relative to the instantaneous position of the particle.

Collecting the terms in eqs. (49) and (50), letting $t_1 \to -\infty$ and (in the regular terms) $t_2 \to t$ one obtains*

$$\left.\begin{aligned}
\mathscr{E}_{\Xi}(t) &= \tfrac{1}{3} Q^2 \frac{4\gamma^2(t_2) - 1}{2(t - t_2)} - Q \int_{-\infty}^{t} \vec{v} \cdot \vec{E}_{in}\, dt \\
&\quad + \tfrac{2}{3} Q^2 \int_{-\infty}^{t} \frac{\vec{g}^{\,2} - (\vec{g} \times \vec{v})^2}{(1 - v^2)^3}\, dt + \mathscr{E}_{in} \\
\vec{\mathscr{P}}_{\Xi}(t) &= \tfrac{2}{3} Q^2 \frac{\gamma^2(t_2)}{(t - t_2)} \vec{v}(t_2) - Q \int_{-\infty}^{t} \{\vec{E}_{in} + \vec{v} \times \vec{H}_{in}\}\, dt \\
&\quad + \tfrac{2}{3} Q^2 \int_{-\infty}^{t} \frac{\vec{g}^{\,2} - (\vec{g} \times \vec{v})^2}{(1 - v^2)^3} \vec{v}\, dt + \vec{\mathscr{P}}_{in}.
\end{aligned}\right\} \quad (51)$$

From this result it is not difficult (see appendix C) to arrive at the desired expressions for the energy and momentum corresponding to the regions Ω and Γ:

$$\left.\begin{aligned}
\mathring{\mathscr{E}}_{\Omega} &= \frac{Q^2}{2\varepsilon} - Q \int_{-\infty}^{t} \vec{v} \cdot \vec{E}_{in}\, dt + \tfrac{2}{3} Q^2 \int_{-\infty}^{t} \frac{\vec{g}^{\,2} - (\vec{g} \times \vec{v})^2}{(1 - v^2)^3}\, dt + \mathscr{E}_{in} \\
\mathring{\vec{\mathscr{P}}}_{\Omega} &= -\tfrac{2}{3} Q^2 \vec{g}(t) - Q \int_{-\infty}^{t} \{\vec{E}_{in} + \vec{v} \times \vec{H}_{in}\}\, dt \\
&\quad + \tfrac{2}{3} Q^2 \int_{-\infty}^{t} \frac{\vec{g}^{\,2} - (\vec{g} \times \vec{v})^2}{(1 - v^2)^3} \vec{v}\, dt + \vec{\mathscr{P}}_{in}
\end{aligned}\right\} \quad (52)$$

and

* Of course we assume that $\gamma^2(t_1)/t_1 \to 0$ for $t_1 \to -\infty$ thus excluding cases like the ideal hyperbolic motion.

$$\left.\begin{aligned}\mathscr{E}_{\Gamma} &= \frac{Q^2}{2\,\varepsilon}\gamma(t)\left(1+\tfrac{1}{3}v^2(t)\right) - Q\int_{-\infty}^{t}\vec{v}\cdot\vec{E}_{in}\,dt \\ &\quad + \tfrac{2}{3}Q^2\int_{-\infty}^{t}\frac{\vec{g}^2-(\vec{g}\times\vec{v})^2}{(1-v^2)^3}\,dt + \mathscr{E}_{in} \\ \vec{\mathscr{P}}_{\Gamma} &= \tfrac{4}{3}\frac{Q^2}{2\,\varepsilon}\gamma(t)\vec{v}(t) - Q\int_{-\infty}^{t}\{\vec{E}_{in}+\vec{v}\times\vec{H}_{in}\}\,dt \\ &\quad + \tfrac{2}{3}Q^2\int_{-\infty}^{t}\frac{\vec{g}^2-(\vec{g}\times\vec{v})^2}{(1-v^2)^3}\vec{v}\,dt + \vec{\mathscr{P}}_{in},\end{aligned}\right\} \tag{53}$$

where, in the latter case, the expansion of $\mathscr{E}_{\Gamma}$ and $\vec{\mathscr{P}}_{\Gamma}$ in powers of ε has been restricted to the leading-order term (proportional to $Q^2/2\varepsilon$) which for dimensional reasons is independent of the acceleration.

Thus, in comparing eq. (52) with eg. (53) written down in the momentary restframe, it should be borne in mind, that in the derivation of eq. (53) from eq. (51) a zeroth-order term in ε, corresponding to a term which, again for dimensional reasons, must be linear in the acceleration, has been neglected.

In passing, it may provoke some reflexion that the Lagrangian density corresponding to the retarded field, $\frac{1}{8\pi}(E_R^2 - H_R^2)$, integrated over the regions $\Xi(t_1 \to -\infty)$ and Γ, respectively, yields the simple results:

$$\frac{1}{8\pi}\int_{\Xi}(E_R^2 - H_R^2)\,dV = \frac{Q^2}{2\,(t-t_2)} \tag{54}$$

$$\frac{1}{8\pi}\int_{\Gamma}(E_R^2 - H_R^2)\,dV = \frac{Q^2}{2\,\varepsilon}\frac{1}{\gamma(t)}. \tag{55}$$

To complete the provisional plan agreed on, it only remains to declare the electromagnetic energy and momentum of the system to be given by that four-vector $\mathscr{P}_{\Omega}(\tau) = (\vec{\mathscr{P}}_{\Omega}(\tau),\, i\mathscr{E}_{\Omega}(\tau))$ which in the momentary rest-frame reduces to $(\mathring{\vec{\mathscr{P}}}_{\Omega},\, i\mathring{\mathscr{E}}_{\Omega})$ given by eq. (52). Thus

$$\left.\begin{aligned}\mathscr{P}_{\Omega}(\tau) &= \frac{Q^2}{2\,\varepsilon}U - \tfrac{2}{3}Q^2 g + \tfrac{2}{3}Q^2\int_{-\infty}^{\tau}g^2 U\,d\tau - Q\int_{-\infty}^{\tau}\mathscr{F}_{(in)}U\,d\tau + \mathscr{P}_{in} \\ &= \int_{-\infty}^{\tau}\left\{\frac{Q^2}{2\,\varepsilon}g - \tfrac{2}{3}Q^2(\dot{g}-g^2U) - Q\mathscr{F}_{(in)}U\right\}d\tau + \mathscr{P}_{in} \\ &= \int_{\not{\Omega}}\mathscr{T}\cdot d\Omega,\end{aligned}\right\} \tag{56}$$

where $\mathscr{T}$ denotes the total electromagnetic energy-momentum tensor, and where the symbol below the integral sign is meant to indicate that the integration is extended over the domain Ω, i.e. that part of the three dimensional hyperplane orthogonal to the momentary four-velocity, which is exterior to a sphere of radius ε centred at the momentary position of the charge. The last step in eq. (56) is justified by noting that in the momentary rest-frame

$$\int_{\diagup\Omega} \mathscr{T} \cdot d\Omega \bigg|_{\text{rest}} = \left(\frac{1}{4\pi} \int_{\Omega} (\vec{E} \times \vec{H}) dV, \;\; \frac{i}{8\pi} \int_{\Omega} (\vec{E}^2 + \vec{H}^2) dV \right) \tag{57}$$

the right-hand side being given by eq. (52).

Compare now eq. (56) with the corresponding expression (not four-vector) within the conventional scheme:

$$(\vec{\mathscr{P}}_{\Gamma}(t), \; i\mathscr{E}_{\Gamma}(t)) = \int_{\overline{\Gamma}} \mathscr{T} \cdot d\Gamma = i \int_{\Gamma} \mathscr{T}_{i4} dV, \;\; (d\Gamma_4 = idV) \tag{58}$$

where the symbol below the integral sign is now meant to indicate that the integration is extended over the domain Γ at constant laboratory time T. Clearly, the important feature to notice is that since in eq. (56) contributions from various spatial regions are added corresponding to simultaneity in the rest-frame (as opposed to eq. (58)), the definition (56) of $\mathscr{P}_{\Omega}(\tau)$ is tantamount to abandoning the idea that the electromagnetic energy and momentum at a given moment is carried by the field at that moment. In this connection it is essential to realize that even though the energy-momentum tensor is divergence-free through all space outside the charge, it is not possible to tilt the cut hyperplane, over which the integral in eq. (56) extends, without changing the value of $\mathscr{P}_{\Omega}(\tau)$. This fact is already manifest from a comparison of the expressions (56) and (53) in the special case where the acceleration actually vanishes at the moment considered.

Although the appearance of the expressions (56) for $\mathscr{P}_{\Omega}(\tau)$ and (53), (58) for $(\mathscr{P}_{\Gamma}, \mathscr{E}_{\Gamma})$ naturally invites comparison with the expressions (41), (45) for $\mathscr{P}_{(\mathring{T})}$ and (40), (44) for $\mathscr{P}(T)$, respectively, it should be borne in mind that, whereas the latter quantities simply refer to different electrified systems, the former ones are competing candidates for the role as the electromagnetic energy and momentum for one definite system, consisting of a point charge interacting with an external field. As a matter of fact, even if Dirac and his school claim to consider the electron strictly as a point charge, nevertheless their exploitation of the conventional Maxwell theory presupposes an underlying picture of the "point electron" as the limit of a tiny charged spherical shell. A further dissimilarity to bear in mind, when comparing the mentioned expressions, arises from the different attitudes towards the stability problem in the two models. In fact, in accordance with the classical Lorentz theory, the stability of the corpuscular system poses no specific problem, once definite assumptions regarding the stabilizing forces are agreed on. In contrast, the very idea behind the endeavours of the proponents of the "point electron" is precisely to avoid any reference to non-electromagnetic forces as stabilizing agents. In this situation the non-vanishing divergence of the electromagnetic energy-momentum tensor becomes an obstacle, which is only circumvented by re-introducing the non-electromagnetic forces well hidden in the disguise of "mass renormalization".

* *

On the background thus acquired it is particularly easy to display the essence of the attempts initiated by Dirac of constructing a renormalized equation of motion for a classical point charge. Indeed, once it has been agreed that the electromagnetic energy and momentum is given by the four-vector eq. (56), the gist of these endeavours amounts — in one way or another — to the assertion of the existence of a conservation law of the form

$$m_b U(\tau) + \mathscr{P}_{\Omega}(\tau) = \text{constant}, \tag{59}$$

where m_b denotes the "bare" mass of the particle. Inserting $\mathscr{P}_{\Omega}(\tau)$ from eq. (56) and differentiating with respect to τ, one immediately arrives at the familiar Lorentz-Dirac equation

$$mg = \tfrac{2}{3} Q^2 [\dot{g} - g^2 U] + Q \mathscr{F}_{(in)} U, \tag{60}$$

where m denotes the "renormalized" mass

$$m = m_b + \frac{Q^2}{2\varepsilon}. \tag{61}$$

Even though the eq. (60) is, of course, known to be approximately valid in many instances, the claim that it represents the exact equation of motion for a classical point charge is unwarranted, in so far as no physical arguments can be adduced neither to justify the identification of the electromagnetic energy and momentum with the components of the four-vector $\mathscr{P}_{\Omega}(\tau)$, (56), nor to support the conservation law (59).

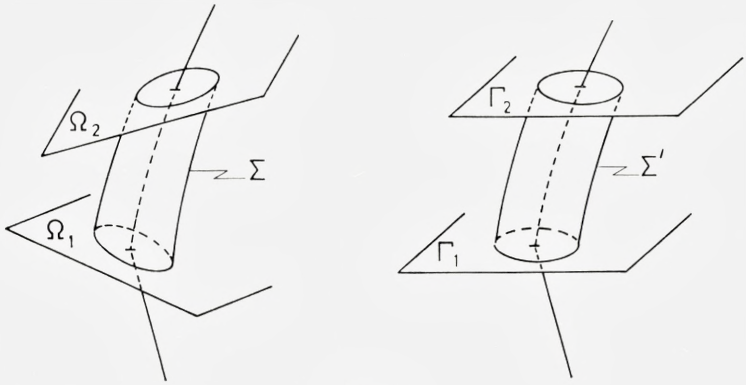

Fig. 4.

After this adumbration of the essential features in the reasoning leading Dirac and others to the "one-body" equation of motion (60), the remainder of this paragraph is devoted to a more careful analysis of the assumptions by which the conventional scheme must be supplemented to allow the deduction of a result, which could not be justified within this frame.

Let us first notice that the difference $\mathscr{P}_\Omega(\tau_2) - \mathscr{P}_\Omega(\tau_1)$ with $\mathscr{P}_\Omega(\tau)$ given by eq. (56), may — by applying Gauss' theorem — be written as

$$\left.\begin{aligned} \mathscr{P}_\Omega(\tau_2) - \mathscr{P}_\Omega(\tau_1) &= \int_{/\Omega_2} \mathscr{T}\cdot d\Omega_2 - \int_{/\Omega_1} \mathscr{T}\cdot d\Omega_1 = \int_\Sigma \mathscr{T}\cdot d\Sigma \qquad &\text{(a)}\\ \int_\Sigma \mathscr{T}\cdot d\Sigma &= \int_{\tau_1}^{\tau_2} d\tau \left\{\frac{Q^2}{2\,\varepsilon} g - \tfrac{2}{3} Q^2(\dot g - g^2 U) - Q\mathscr{F}_{(in)} U\right\}, \qquad &\text{(b)} \end{aligned}\right\} \quad (62)$$

where Σ denotes a 3-dimensional tube surrounding the world line bound by the two cut planes Ω_1 and Ω_2 and where the intersection between the tube and the mentioned planes is the two-dimensional spherical surface of radius ε (see figure 4). The identity (62 b), which forms the basis for Dirac's discussion is derived by him through direct expansion of the energy-momentum tensor in powers of ε.

A critical step in Dirac's analysis is his identification of the left hand side of eq. (62) as "the difference in energy (or momentum) residing within the tube at the two ends...". Indeed, implicitly relying on the assumption that the energy and momentum within the tube at the two ends constitute a four-vector, Dirac demands that this four-vector at any given point $x(\tau)$ of the world line be expressible as some universal function B of the particle variables $(U, \dot U, \ddot U, \ldots)$ at that point. Consequently, it is required that

$$B(U(\tau_2), \dot{U}(\tau_2), \ldots) - B(U(\tau_1), \dot{U}(\tau_1), \ldots) = \int_{\Sigma} \mathcal{T} \cdot d\Sigma. \qquad (63)$$

Clearly, the integral (62b) extended along an arbitrary world line would not in general possess this remarkable property, which, of course, amounts to requiring that the integrand in eq. (62b) be equal to $\dot{B}$, i.e., the differential of the universal function B. Hence, the demand (63) is but a recipe for the selection of the set of *permissible* world lines, and thus — for each choice of B — it is equivalent to *prescribing* the equations of motion for the particle.

Returning to the above-mentioned crucial interpretation of the left-hand side of eq. (62), we have just seen that the four-vector character of this integral necessitates the assumption that the energy and momentum within the tube constitute a four-vector. However, since the energy and momentum within the tube regarded as integrals over appropriate densities at definite laboratory time do not form a four-vector, Dirac's identification amounts to a re-definition of these quantities, analogous to that discussed above for the energy and momentum "outside" the tube*. In particular, combining the eqs. (62a) and (63), we see that the conservation of the total four-momentum is expressed** as

$$\mathcal{P}_{\Omega}(\tau) - B(\tau) = \text{constant}. \qquad (64)$$

In contrast to what is the case in the conventional scheme, this equation implies that the conserved four-momentum for the total closed system can be decomposed into a sum of four-momenta referring to the interacting subsystems.

It is instructive to paraphrase the above "deductions" within the conventional scheme. Here the difference in electromagnetic energy and momentum "outside" the tube at two successive instances T_1 and T_2, referred to one and same system of inertia, is given by:

$$\left.\begin{array}{l} \left.\begin{array}{l} (\vec{\mathcal{P}}_{\Gamma}(T_2), i\mathcal{E}_{\Gamma}(T_2)) - (\vec{\mathcal{P}}_{\Gamma}(T_1), i\mathcal{E}_{\Gamma}(T_1)) \\ = \int\limits_{\overline{\Gamma_2}} \mathcal{T} \cdot d\Gamma_2 - \int\limits_{\overline{\Gamma_1}} \mathcal{T} \cdot d\Gamma_1 = \int_{\Sigma'} \mathcal{T} \cdot d\Sigma' \end{array}\right\} \text{(a)} \end{array}\right\} \qquad (65)$$

* This remark is further substantiated by the observation that the integral over the surface Σ, referred to by Dirac as the flow of energy and momentum through the tube, cannot in general – when the end-surfaces Ω_1 and Ω_2 are tilted relative to each other – be interpreted as the flux through a moving surface during a *definite* time interval.

** Dirac remarks that the simplest choice for B would be $B = -m_b U$ in which case the conservation laws (64) and (59) become identical.

$$\left.\left.\begin{aligned}\int_{\Sigma'} \mathscr{T} \cdot d\Sigma' &= \frac{Q^2}{2\varepsilon}\gamma(T_2)\left(\tfrac{4}{3}\vec{v}(T_2),\, i(1+\tfrac{1}{3}\vec{v}^2(T_2))\right)\\ &- \frac{Q^2}{2\varepsilon}\gamma(T_1)\left(\tfrac{4}{3}\vec{v}(T_1),\, i(1+\tfrac{1}{3}\vec{v}^2(T_1))\right)\\ &- \int_{T_1}^{T_2} d\tau\{-\tfrac{2}{3}Q^2 g^2 U + Q\mathscr{F}_{(in)}U\},\end{aligned}\right\}\text{(b)}\right\} \quad (65)$$

where Γ_1 and Γ_2 denote the cut hyperplanes T_1 = const. and T_2 = const., respectively, and where Σ' refers to the surface of the tube between Γ_1 and Γ_2, the intersection between the tube and the planes Γ_1 and Γ_2 being Heaviside ellipsoids (see figure 4). Furthermore, we have used eq. (53) neglecting again in the expressions for $\mathscr{E}_\Gamma$ and $\vec{\mathscr{P}}_\Gamma$ a term linear in the acceleration*. Insisting that the permissible world lines are selected according to the requirement that the left-hand side of eq. (65b) be expressible as the difference in values taken by some universal function $B' = (\vec{B}'(T), iB_0'(T))$ (not four-vector) of the particle variables at times T_2 and T_1, we have now in place of eq. (63)

$$B'(\vec{v}(T_2), \vec{g}(T_2), \ldots) - B'(\vec{v}(T_1), \vec{g}(T_1), \ldots) = \int_{\Sigma'} \mathscr{T} \cdot d\Sigma', \quad (66)$$

which combined with eq. (65a) leads to conservation law (analogous to eq. (64))

$$(\vec{\mathscr{P}}_\Gamma(T), \mathscr{E}_\Gamma(T)) - (\vec{B}'(T), B_0'(T)) = \text{constant}. \quad (67)$$

In this case, the energy and momentum of the subsystems, adding up to the total conserved four-momentum, do not themselves constitute the components of four-vectors. Nevertheles, it is still possible to perform a mass renormalization, and the form of eq. (53) immediately suggests that the most alluring** choice of B' would be

$$(\vec{B}', iB_0') = -m_b U + \frac{Q^2}{2\varepsilon}\gamma\left(\tfrac{1}{3}\vec{v}, i\tfrac{1}{3}v^2\right) + \tfrac{2}{3}Q^2 g \quad (68)$$

where m_b again denotes the bare mass. Indeed, combining the eqs. (53), (67) and (68), one arrives once more at the conservation law

* It is easy to verify that the difference between the right-hand sides of eqs. (62b) and (65b) just equals the flux of $\mathscr{T}$ through the tiny sections of the tube between the planes Ω_1 and Γ_1, and Ω_2, and Γ_2.

** It should be noted, that the four functions B' cannot be chosen completely independently of each other, since we are dealing with only three independent equations of motion. Thus, the choice of the first two members on the right hand side of eq. (68) immediately implies the need for a third term to ensure the mutual compatibility of the resulting equations of motion, and it is easily seen, that $\frac{2}{3}Q^2 g$ represents the simplest possible choice for this additional term.

$$mU(\tau) - \int_{-\infty}^{\tau} d\tau \{ \tfrac{2}{3} Q^2 (\dot{g} - g^2 U) + Q \mathscr{F}_{(in)} U \} = \text{constant}, \tag{69}$$

where the renormalized mass m is again defined by eq. (61). Differentiation of eq. (69) immediately gives back the equations of motion (60).

From the above "deductions" it emerges that the most prominent departure by Dirac and his followers from the conventional scheme — namely, the re-definition of the electromagnetic energy and momentum so as to transform like a four-vector — surprisingly enough turns out to be unessential, at least in so far as the resulting equation of motion does not depend on this assumption. Instead, the pivot, on which the entire argumentation turns, is seen to be the much less conspicuous step of taking for granted that the energy and momentum "residing within the tube at the two ends" should be a state function expressible solely in terms of the particle variables*. It is through the insistence that this demand on the integral (62) be the guiding principle for the selection of the permissible world lines that the ground of classical electrodynamics is left behind. Indeed, this "principle" merely conceals a postulate of the desired equation of motion.

§ 4. Concluding Remarks

As emphasized in the preceding discussion, Classical Electron Theory does provide a well-defined framework within which any question, concerning the behaviour of electrified bodies, which may at all be formulated in terms of classical physical ideas, can in principle be answered, irrespectively of the magnitude of the charge and mass of the bodies concerned. In contrast, as analysed in the previous paragraph, the attempts by Dirac, Rohrlich and others to implement the scheme of classical electrodynamics have not resulted in a systematic description in which the notion of a point charge is harmoniously incorporated into the ordinary Maxwell theory for extended charge distributions. Furthermore, the physical interpretation of the new scheme is hampered by the well-known difficulties associated with the appearance of "advanced effects" or "acausalities" in the solutions of the Lorentz-Dirac equation. From the conventional standpoint these difficulties may be explained simply as the result of an unwarranted extrapolation of conclusions drawn on the basis of an *approximate* equation of motion. How-

* It is even more misleading when some authors profess to "derive" this property by arguing that the integral (62) is independent of the shape of the *tube*. Of course, the question at issue concerns a variation of the world line.

ever, since in the "point electron theory" the Lorentz-Dirac equation is considered an *exact* equation of motion, the "acausalities" acquire a fundamental status thereby creating the need for a comprehensive revision of the conceptual framework. Indeed, it seems that the prediction of "advanced effects" in the theory represents a contradiction in terms unless it is explicitly assumed that – for some reason or another – the freedom, commonly assumed, of external agents or "observers" to intervene in the system under consideration is limited. As long as this feature is not reflected in the formal description itself – in the way the reciprocal measuring limitations are built into the foundations of quantum theory –, the description remains logically incomplete and the question as to its observable consequences cannot even be formulated, much less answered*.

It needs hardly be added that a solution to the logical dilemma represented by the prediction of "acausalities" cannot be achieved by reference to the empirical limitations of the classical description itself. Indeed, in any comparison between the "point electron theory" and the conventional scheme, it is of course essential as clearly as possible to distinguish between the problem of internal consistency of the description on the one hand, and the question of its range of empirical validity on the other. In the present context, reference to empirical evidence merely serves to emphasize that since quantum phenomena become important already when probing into regions of extension far bigger than the classical electron radius, there is – empirically speaking – no room for unambiguous application of a "point electron theory" within classical physics.

Another aspect of the problems discussed is associated with the consequent use of the concept of "radiation" within the classical description. In § 2 a simple example was analysed, which exhibited the inadequacy of attempts to picture the act of radiation emission as a continuous process localized in space and time, and it was concluded that the radiation process entailed a modification of the electromagnetic field as a whole. Thus, the fact that the field strength in a given space-time domain is causally connected to the motion of the source particle at a definite segment of the world line, does not provide a physical basis for the notion that the field energy associated with the domain considered has been "emitted" by the charge on the corresponding segment.

Clearly, the above conclusions bring in relief the arbitrariness, discussed

* It has been suggested (ROHRLICH, loc. cit.), that detection of radiation from a uniformly accelerated electron should provide direct evidence for an acausal equation of motion. It will be clear from part III of these studies that this idea cannot be upheld.

in § 3, in the interpretation of the formal expressions for the "energy-momentum flux" through the tube surrounding the world line of the charge. Particularly misleading in this context is the occasional reference in the literature to an analogy with "photons", since the very definition of this concept excludes any well-defined application of the field picture, on which the entire discussion is based.

Thus, in dealing with radiation phenomena, we are presented with a feature of wholeness, familiar in quantal processes, but less so within the domain of classical physics. This feature receives a particular emphasis in the "action-at-a-distance" formulation of classical electrodynamics by FEYNMAN and WHEELER[2]), who, however, by completely eliminating the degrees of freedom associated with the field, are led to give up the notion of instantaneous energy-momentum balance, at least in its customary form.

Among the attempts to formulate a classical theory of a "point electron" the work of FEYNMAN and WHEELER is distinguished by its inner consequence. As already discussed, in a "point electron" theory the presence of damping poses a problem without counterpart in the conventional scheme. In fact, within the former description the damping must either be considered the result of the action of the self-field on the particle, through a mechanism which, however, by the very idea of a point charge remains unanalysible, thus reducing the problem at issue to being a matter of composing a recipe for evaluating the effect. Or, more consequently relative to the premises, the possibility of self-interactions is denied altogether, in conformity with the conception of "charge" as an elementary property of the particle, expressing its ability to influence other similar particles according to a definite set of rules. Thus, in this case the presence of the damping acting on an individual particle can only be related to the interaction with other distant particles outside the system under consideration, and this interaction cannot possibly be retarded, even if only an approximate simultaneity between the motion and the damping of the particle is insisted upon.

A solution to this problem was achieved by FEYNMAN and WHEELER through the introduction of an allegedly "fundamental" time symmetric interaction, which, on the one hand, makes possible the description of the damping acting on the individual particle as the advanced effect of the polarization induced in the "distant absorbers" by the retarded interaction generated by the particle, and which, on the other hand — through a subtle interference between the advanced fields of the absorber and the charge, eliminating all advanced effects prior to the motion of the source — guarantees the reproduction of the usual "effectively" retarded field as generated by the charge in question.

Notwithstanding the lesson conveyed by the very possibility of constructing a coherent description of electrodynamics so far beyond immediate conceptions, it remains a delicate question to what extent the scheme admits of the intervention of external irreversible devices, not necessarily of electromagnetic origin. Indeed, in the absence of absorbers the time symmetric scheme is clearly incompatible with the presence of irreversibly functioning contrivances capable of distinguishing future from past and hence also of making a non-predictable choice as to whether or not to prevent the occurrence of an event, which in the description is held responsible for actions already completed. Such paradoxes are avoided by the introduction of the distant absorbers, which, as already indicated, causes the interaction to become "effectively" retarded. However, in spite of the apparent formal unimpeachability achieved through the above-mentioned destructive interference between the advanced fields, crucial for the compatibility of the time symmetric scheme and the possibility of influencing the future, there seems to be an inherent ambiguity in the notion of an advanced field — existing at all times prior to the future event to which it is correlated — which again becomes especially conspicuous if the occurrence of the mentioned event is made dependent on the outcome of a process which is unpredictable in principle.

As is evident from these considerations, not only the need but also the room for transcending classical electrodynamics, as proposed by Feynman and Wheeler, is procured just by the new element added to it, namely the idea of an indivisible point charge. In fact, as far as observable consequences are concerned, the new scheme reduces, in the case of complete absorption, identically to the conventional electrodynamics, if this concept is abandoned, the damping becoming again an expression for the mutual interaction between the infinitesimal constituents of the "point charge".

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Appendix A

Evaluation of the difference $\mathscr{F}_R(x) - \mathscr{F}_A(x)$ in the vicinity of the world line.

The difference between the retarded and advanced Lienard-Wiechert potentials at a fixed space-time point x is given by

$$A_R(x) - A_A(x) = -\delta q \left\{ \frac{U(\tau)}{(x-x(\tau))U(\tau)}\bigg|_{\tau_R} + \frac{U(\tau)}{(x-x(\tau))U(\tau)}\bigg|_{\tau_A} \right\}, \tag{A1}$$

where $\tau_A > \tau_R$ denote the two roots of the equation $(x-x(\tau))^2 = 0$. Expanding the function $(x-x(\tau))^2$ around its extremal value ε^2, which for convenience is taken to occur for $\tau = 0$, introducing the abreviation $l = x - x(0)$, and agreeing that all quantities U, g, $\dot{g}$ etc. written without argument, refer to the value $\tau = 0$, one has

$$\left.\begin{aligned} (x-x(\tau))^2 &= \varepsilon^2 - (1+gl)\tau^2 - \tfrac{1}{3}(\dot{g}l)\tau^3 - \tfrac{1}{12}g^2\tau^4 + \dots \\ (x-x(\tau))\,U(\tau) &= (1+gl)\tau + \tfrac{1}{2}(\dot{g}l)\tau^2 + \tfrac{1}{6}g^2\tau^3 + \dots \end{aligned}\right\} \tag{A2}$$

Hence to the accuracy required, the sum and the product of the roots of the equation $(x-x(\tau))^2 = 0$ are given by

$$\tau_R + \tau_A \simeq -\tfrac{1}{3}(\dot{g}l)\varepsilon^2, \qquad \tau_R\tau_A \simeq -\varepsilon^2. \tag{A3}$$

Next, expand the difference (A1) in the form

$$\left.\begin{aligned} A_R(x) - A_A(x) &= \delta q \left\{ \left(\frac{a_{-1}}{\tau_R} + a_0 + a_1\tau_R + \dots \right) + \left(\frac{a_{-1}}{\tau_A} + a_0 + a_1\tau_A + \right) \right\} \\ &= \delta q \left\{ a_{-1}\frac{\tau_R+\tau_A}{\tau_R\tau_A} + 2\,a_0 \right\} + o(\varepsilon^2), \end{aligned}\right\} \tag{A4}$$

where the constant vectors a_1, and a_0 are seen to be determined by

$$a_{-1} = \frac{U}{1+gl}, \quad a_0 = \frac{d}{d\tau}\frac{\tau U(\tau)}{(x-x(\tau))U(\tau)}\bigg|_{\tau=0} = \frac{g}{1+gl} - \tfrac{1}{2}(\dot{g}l)U. \tag{A5}$$

Finally, remembering that U, g, $\dot{g}$ etc. depend implicitly on x, one has

$$\left.\begin{aligned} A_R(x) - A_A(x) &= -\delta q \left\{ \frac{2g}{1+gl} - \tfrac{2}{3}(\dot{g}l)\,U \right\} + o(\varepsilon^2) \\ &= -\delta q \left\{ \tfrac{4}{3}(\dot{g}l)\,U + 2\partial_x ln(1+gl) \right\} + o(\varepsilon^2). \end{aligned}\right\} \tag{A6}$$

Whence the field tensor, correct to zeroth order in ε, is obtained as

$$\mathscr{F}_R(x) - \mathscr{F}_A(x) = \partial_x \wedge (A_R(x) - A_A(x)) \simeq -\tfrac{4}{3}\delta q \partial \wedge (\dot{g}l) U \simeq \tfrac{4}{3}\delta q U \wedge \dot{g}, \quad \text{(A7)}$$

where it has been observed, that a change δx in x causes a change $\delta l = \delta x + (U\delta x) U$ in l.

Appendix B

Evaluation of energy-momentum expenditure to bring two electrified corpuscles from rest at infinite separation along arbitrary world-lines to a state of common uniform motion.

To evaluate the integral $\mathscr{I}$ in eq. (36), introduce the retarded Green's function $\mathscr{D}_R(x)$:

$$\left.\begin{aligned} \mathscr{I} &= \int^T dx \{\mathscr{F}_R^{(2)}(x) s_1(x) + \mathscr{F}_R^{(1)}(x) s_2(x)\} \\ &= \iint^T dx dy \{[\partial_x \mathscr{D}_R(x-y) \wedge s_2(y)] s_1(x) + [\partial_x \mathscr{D}_R(x-y) \wedge s_1(y)] s_2(x)\} \\ &\equiv \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3, \end{aligned}\right\} \quad \text{(B1)}$$

where

$$\left.\begin{aligned} \mathscr{I}_1 &= \iint^T dx dy (\partial_x \mathscr{D}_R(x-y)) s_2(y) \cdot s_1(x) \\ \mathscr{I}_2 &= \iint^T dx dy (\partial_x \mathscr{D}_R(x-y)) s_1(y) \cdot s_2(x) \\ \mathscr{I}_3 &= -\iint^T dx dy \{(s_1(x) \cdot \partial_x \mathscr{D}_R(x-y)) s_2(y) + (s_2(x) \cdot \partial_x \mathscr{D}_R(x-y)) s_1(y)\}. \end{aligned}\right\} \quad \text{(B2)}$$

The integral $\mathscr{I}_3$ is immediately evaluated by partial integration and application of the equation of continuity for the current densities. Taking into account the boundary conditions at $t = -\infty$ one obtains

$$\mathscr{I}_3 = -\delta q_1 A_R^{(2)}(\vec{x}_1(T), T) - \delta q_2 A_R^{(1)}(\vec{x}_2(T), T) = -2 \frac{\delta q_1 \delta q_2}{\varepsilon} U, \quad \text{(B3)}$$

where the last equality is justified by the assumption stated in the text, that the motion has been uniform for at least a time $2\gamma\varepsilon$ prior to T.

Consider next the term $\mathscr{I}_2$, remembering the symmetry relation $\mathscr{D}_R(x) = \mathscr{D}_A(-x)$:

$$\left.\begin{aligned}\mathscr{I}_2 &= \iint dxdy\,\Theta(T-t_x)(\partial_x \mathscr{D}_R(x-y))s_1(y)\cdot s_2(x)\\ &= -\iint dxdy\,\Theta(T-t_x)(\partial_y \mathscr{D}_A(y-x))s_2(x)\cdot s_1(y)\\ &= -\int dy\Big[\partial_y \int dx\,\Theta(T-t_x)\mathscr{D}_A(y-x)s_2(x)\Big]s_1(y)\\ &= -\int dy\Big[\partial_y\,\Theta(T-t_y-|\vec{x}_2(T)-\vec{y}\,|)A_A^{(2)}(y)\Big]s_1(y),\end{aligned}\right\}\quad \text{(B4)}$$

where $\Theta(t)$ denotes the step function

$$\Theta(t) = \begin{cases}1 & \text{for} \quad t>0\\ 0 & \text{for} \quad t<0\end{cases}\quad \text{(B5)}$$

obeing the relation

$$\partial_x \Theta(t-|\vec{x}\,|) = -2\Theta(t)\delta(x^2)x. \quad \text{(B6)}$$

Thus one obtains

$$\left.\begin{aligned}\mathscr{I}_2 = &-\int dy\,\Theta(T-t_y-|\vec{x}_2(T)-\vec{y}\,|)[\partial_y A_A^{(2)}(y)]s_1(y)\\ &-\int dy\,2\Theta(T-t_y)\delta((x_2-y)^2)(A_A^{(2)}(y)\cdot s_1(y))(x_2-y)\\ = &-\int^{t_{12}} dy[\partial_y A_A^{(2)}(y)]s_1(y)+\frac{\delta q_1\,\delta q_2}{\varepsilon^2}R^{(21)},\end{aligned}\right\}\quad \text{(B7)}$$

where $x_2 = (\vec{x}_2(T), iT)$ and where t_{12} and the light-vector $R^{(21)}$ are defined on figure 2 of the text.

From the results (B1), (B2), (B3) and (B7) one finds:

$$\left.\begin{aligned}\mathscr{I} = &\int^{t_{12}} dx[\partial_x(A_R^{(2)}-A_A^{(2)})]s_1(x)+\frac{\delta q_1\,\delta q_2}{\varepsilon^2}l^{(21)}-\frac{\delta q_1\,\delta q_2}{\varepsilon}U\\ &+\int_{t_{12}}^{T} dx[\partial_x A_R^{(2)}(x)]s_1(x).\end{aligned}\right\}\quad \text{(B8)}$$

Remembering that for the time interval over which the last integral extends

$$A_R^{(2)}(x_1(\tau)) = \delta q_2\frac{U}{\varepsilon} \quad \text{and} \quad \partial_x\varepsilon = -\frac{l^{(21)}}{\varepsilon},$$

the last integral in eq. (B8) is evaluated to yield

$$-\frac{\delta q_1\,\delta q_2}{\varepsilon^3}l^{(21)}\frac{(T-t_{12})}{\gamma}, \quad \text{(B9)}$$

where

$$T - t_{12} = \frac{1}{i} R_4^{(21)} = \frac{1}{i}(l_4^{(21)} + i\varepsilon\gamma).$$

Subtracting finally the quantity

$$\int^{t_{12}} dx\, s_1(x) \cdot \partial_x [A_R^{(2)}(x) - A_A^{(2)}(x)] = \int^{t_{12}} dx\, \partial_x \cdot [s_1(x)(A_R^{(2)} - A_A^{(2)})] = 0$$

(remembering that $A_R = A_A$ for the final rectilinear sections of the world lines) one obtains

$$\mathscr{I} = \int^{t_{12}} dx [\partial_x \wedge (A_R^{(2)} - A_A^{(2)})] s_1(x) - \frac{\delta q_1 \delta q_2}{\varepsilon} U - \frac{\delta q_1 \delta q_2}{\varepsilon^3} \begin{pmatrix} l_4^{(21)} \\ i\gamma \end{pmatrix} l^{(21)}. \tag{B10}$$

Appendix C

Explicit evaluation of the energy and momentum associated with the domains Ξ, Γ and Ω.

By means of the expression for $\vec{E}_{\mathrm{I}}$, $\vec{H}_{\mathrm{I}}$, $\vec{E}_{\mathrm{II}}$, $\vec{H}_{\mathrm{II}}$ as defined by eq. (47) one finds

$$\left.\begin{aligned} &\frac{1}{8\pi}\int_{\Xi} dV [\vec{E}_{\mathrm{I}}^2 + 2\vec{E}_{\mathrm{I}} \cdot \vec{E}_{\mathrm{II}} + \vec{H}_{\mathrm{I}}^2 + 2\vec{H}_{\mathrm{I}} \cdot \vec{H}_{\mathrm{II}}] = \\ &\frac{1}{8\pi}\int_{\Xi} dV \left\{ \frac{(1-v^2)^4}{(t-t_R)^4} \frac{1+v^2-(\vec{v}\cdot\hat{n})^2 - 2\vec{v}\cdot\hat{n}}{(1-\vec{v}\cdot\hat{n})^6} \right. \\ &\quad \left. + 4\frac{(1-v^2)}{(t-t_R)^3}\left[\frac{\vec{v}\cdot\vec{g}}{(1-\vec{v}\cdot\hat{n})^5} - \frac{(\hat{n}\cdot\vec{v})(\hat{n}\cdot\vec{g})}{(1-\vec{v}\cdot\hat{n})^6} + \frac{\hat{n}\cdot\vec{g}}{(1-\vec{v}\cdot\hat{n})^6} v^2\right]\right\}. \end{aligned}\right\} \tag{C1}$$

$$\left.\begin{aligned} &\frac{1}{8\pi}\int_{\Xi} dV [\vec{E}_{\mathrm{II}}^2 + \vec{H}_{\mathrm{II}}^2] = \\ &\frac{1}{8\pi}\int_{\Xi} dV \left\{ \frac{g^2}{(1-\vec{v}\cdot\hat{n})^4} + \frac{(v^2-1)(\hat{n}\cdot g)^2}{(1-\vec{v}\cdot\hat{n})^6} + \frac{2(\hat{n}\cdot\vec{g})(\vec{v}\cdot\vec{g})}{(1-\vec{v}\cdot\hat{n})^5} \right\} \frac{1}{(t-t_R)^2} \end{aligned}\right\} \tag{C2}$$

$$\left.\begin{aligned} &\frac{1}{4\pi}\int_{\Xi} dV [\vec{E}_{\mathrm{I}} \times \vec{H}_{\mathrm{I}} + \vec{E}_{\mathrm{I}} \times \vec{H}_{\mathrm{II}} + \vec{E}_{\mathrm{II}} \times \vec{H}_{\mathrm{I}}] = \\ &\frac{1}{4\pi}\int_{\Xi} dV \left\{ \left[\frac{1-v^2}{(1-\vec{v}\cdot\hat{n})^6} \frac{1+v^2-2\vec{v}\cdot\hat{n}}{(t-t_R)^4} \right.\right. \end{aligned}\right\} \tag{C3}$$

$$
\left.\begin{aligned}
&+\frac{2}{(t-t_R)^2}\frac{1-v^2}{(1-\vec{v}\cdot\hat{n})^6}\left[(\vec{v}\cdot\vec{g})(1-\vec{v}\cdot\hat{n})-(\hat{n}\cdot\vec{v})(n\cdot\vec{g})+v^2\hat{n}\cdot\vec{g}\right]\hat{n}\\
&-\frac{(1-v^2)}{(t-t_R)^2(1-\vec{v}\cdot\hat{n})^2}\left[\frac{1-v^2}{(1-\vec{v}\cdot\hat{n})^3}\frac{\hat{n}-\vec{v}}{(t-t_R)^2}+\frac{(\hat{n}\cdot\vec{g})(\hat{n}-\vec{v})-(1-\hat{n}\cdot\vec{v})\vec{g}}{(1-\vec{v}\cdot\hat{n})^3(t-t_R)}\right]\Bigg\}.
\end{aligned}\right\} \quad \text{(C3)}
$$

$$
\left.\begin{aligned}
&\frac{1}{4\pi}\int_{\Xi} dV(\vec{E}_{\mathrm{II}}\times\vec{H}_{\mathrm{II}}) =\\
&\frac{1}{4\pi}\int_{\Xi} dV\left\{\frac{\vec{g}^2}{(1-\hat{n}\cdot\vec{v})^4}+\frac{(v^2-1)(\hat{n}\cdot\vec{g})^2}{(1-\vec{v}\cdot\hat{n})^6}+\frac{2(\hat{n}\cdot\vec{g})(\vec{v}\cdot\vec{g})}{(1-\vec{v}\cdot\hat{n})^5}\right\}\frac{\hat{n}}{(t-t_R)^2}.
\end{aligned}\right\} \quad \text{(C4)}
$$

To find the value of these integrals at time t, each point $\vec{x}$ in the domain Ξ is parametrized by the corresponding retarded point $\vec{x}_R(t_R)$ on the particle trajectory. In terms of the variables $\hat{n} = \bar{x}-\bar{x}_R/|\bar{x}-\bar{x}_R|$ and $t-t_R$, the volume element is easily seen to be given by*

$$
dV = (1-\vec{v}_R\cdot\hat{n})(t-t_R)^2 dt_R\, d\Omega \qquad \text{(C5)}
$$

and the outer (inner) boundary of Ξ to be determined by $t_R = t_1 (t_R = t_2)$. Differentiating the identities

$$
\int\frac{d\Omega}{4\pi}\frac{1}{(1-\vec{v}\cdot\hat{n})^2} = \frac{1}{1-v^2} \quad\text{and}\quad \int\frac{d\Omega}{4\pi}\frac{1}{(1-\vec{v}\cdot\hat{n})^3} = \frac{1}{(1-v^2)^2}
$$

the appropriate number of times with respect to the components of $\vec{v}$, the necessary angular integrals are immediately obtained:

$$
\int\frac{d\Omega}{4\pi}\frac{n_\iota}{(1-\vec{v}\cdot\hat{n})^3} = \frac{v_\iota}{(1-v^2)^3} \qquad \int\frac{d\Omega}{4\pi}\frac{n_\iota}{(1-\vec{v}\cdot\hat{n})^4} = \frac{4}{3}\frac{v_\iota}{(1-v^2)^3}
$$

$$
\int\frac{d\Omega}{4\pi}\frac{n_\iota n_\varkappa}{(1-\vec{v}\cdot\hat{n})^4} = \frac{4}{3}\frac{v_\iota v_\varkappa}{(1-v^2)^3}+\frac{1}{3}\frac{\delta_{\iota\varkappa}}{(1-v^2)^2}
$$

$$
\int\frac{d\Omega}{4\pi}\frac{n_\iota n_\varkappa}{(1-\vec{v}\cdot\hat{n})^5} = 2\frac{v_\iota v_\varkappa}{(1-v^2)^4}+\frac{1}{3}\frac{\delta_{\iota\varkappa}}{(1-v^2)^3}
$$

$$
\int\frac{d\Omega}{4\pi}\frac{n_\iota n_\varkappa n_\lambda}{(1-\vec{v}\cdot\hat{n})^5} = 2\frac{v_\iota v_\varkappa v_\lambda}{(1-v^2)^4}+\frac{1}{3}\frac{v_\iota\delta_{\varkappa\lambda}+v_\lambda\delta_{\iota\varkappa}+v_\varkappa\delta_{\lambda\iota}}{(1-v^2)^3}.
$$

* The use of the standard notation $d\Omega$ for the surface element on the unit sphere should here cause no confusion with the volume element on the hyperplane Ω used elsewhere.

Inserting these integrals and the volume element into eqs. (C1)–(C4), one finds the results (49a, b) and (50a, b) of the text:

$$\left.\begin{aligned}&\frac{1}{8\pi}\int_{\Xi}[\vec{E}_{\mathrm{I}}^2+2\vec{E}_{\mathrm{I}}\cdot\vec{E}_{\mathrm{II}}+\vec{H}_{\mathrm{I}}^2+2\vec{H}_{\mathrm{I}}\cdot\vec{H}_{\mathrm{II}}]\,dV\\&=\frac{Q^2}{2}\int_{t_1}^{t_2}dt_R\left\{\frac{1}{(t-t_R)^2}\frac{1+v^2/3}{1-v^2}+\tfrac{8}{3}\frac{1}{(t-t_R)}\frac{\vec{v}\cdot\vec{g}}{(1-v^2)}\right\}\\&=\frac{Q^2}{3}\left\{\frac{4\gamma^2(t_2)-1}{2(t-t_2)}-\frac{4\gamma^2(t_1)-1}{2(t-t_1)}\right\}\end{aligned}\right\}\quad(49\,\mathrm{a})$$

$$\frac{1}{8\pi}\int_{\Xi}[\vec{E}_{\mathrm{II}}^2+\vec{H}_{\mathrm{II}}^2]\,dV=\tfrac{2}{3}Q^2\int_{t_1}^{t_2}dt_R\frac{\vec{g}^2-(\vec{g}\times\vec{v})^2}{(1-v^2)^3}\quad(49\,\mathrm{b})$$

$$\left.\begin{aligned}&\frac{1}{4\pi}\int_{\Xi}[\vec{E}_{\mathrm{I}}\times\vec{H}_{\mathrm{I}}+\vec{E}_{\mathrm{I}}\times\vec{H}_{\mathrm{II}}+\vec{E}_{\mathrm{II}}\times\vec{H}_{\mathrm{I}}]\,dV\\&=Q^2\int_{t_1}^{t_2}dt_R\left\{\tfrac{2}{3}\frac{\vec{v}}{(1-v^2)(t-t_R)^2}+\tfrac{2}{3}\frac{\vec{g}}{(t-t_R)(1-v^2)}+\tfrac{4}{3}\frac{(\vec{g}\cdot\vec{v})\vec{v}}{(t-t_R)(1-v^2)^2}\right\}\\&=\tfrac{2}{3}Q^2\left\{\frac{\gamma^2(t_2)}{(t-t_2)}\vec{v}(t_2)-\frac{\gamma^2(t_1)}{(t-t_1)}\vec{v}(t_1)\right\}.\end{aligned}\right\}\quad(50\,\mathrm{a})$$

$$\frac{1}{4\pi}\int_{\Xi}[\vec{E}_{\mathrm{II}}\times\vec{H}_{\mathrm{II}}]\,dV=\tfrac{2}{3}Q^2\int_{t_1}^{t_2}dt_R\frac{\vec{g}^2-(\vec{g}\times\vec{v})^2}{(1-v^2)^3}\vec{v}.\quad(50\,\mathrm{b})$$

* * *

To derive the equation (50d), notice, that $\vec{E}_R$, $\vec{H}_R$ satisfy the Maxwell equations with the currents $[\vec{j}(\vec{x}),\,i\varrho(\vec{x})]=\delta(\vec{x}-\vec{x}(t))\,[\vec{v},\,i]$, whereas $\vec{E}_{in}$, $\vec{H}_{in}$ satisfy the free Maxwell equations. Thus

$$Q\int_{-\infty}^{t}dt[E_{in}(\vec{x}(t))+\vec{v}\times\vec{H}_{in}(\vec{x}(t))]=\int_{-\infty}^{t}dt\int dV[\vec{E}_{in}(\vec{x})\varrho(\vec{x})+\vec{j}(\vec{x})\times\vec{H}_{in}(\vec{x})]$$
$$=\frac{1}{4\pi}\int_{-\infty}^{t}dt\int dV\left[\vec{E}_{in}\,\mathrm{div}\,\vec{E}_R-\vec{H}_{in}\times\mathrm{rot}\,\vec{H}_R+\vec{H}_{in}\times\frac{\partial\vec{E}_R}{\partial t}\right].$$

From the identity

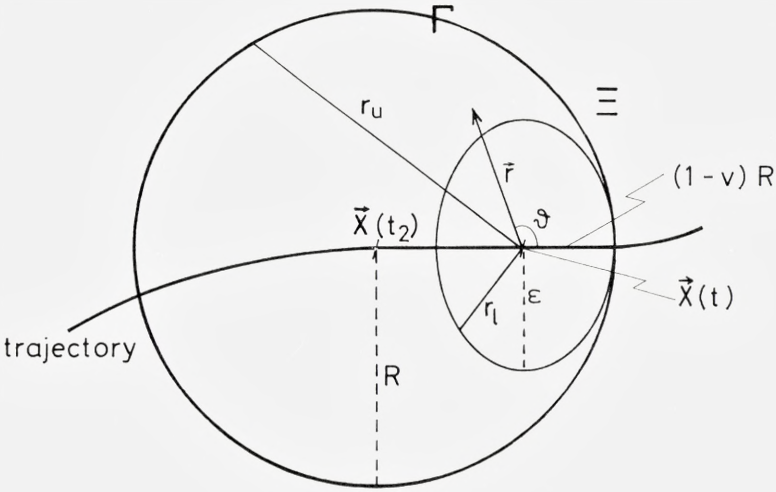

Fig. 5.

$$(\vec{A}\ \mathrm{div}\ \vec{B} + \vec{B}\ \mathrm{div}\ \vec{A})_\iota = (\vec{A} \times \mathrm{rot}\ \vec{B} + \vec{B} \times \mathrm{rot}\ \vec{A})_\iota + \partial_\varkappa (A_\iota B_\varkappa + A_\varkappa B_\iota - \delta_{\iota\varkappa} \vec{A} \cdot \vec{B})$$

valid for any two vector fields $\vec{A}$ and $\vec{B}$, we find with $\vec{A} = \vec{E}_{in}$ and $\vec{B} = \vec{E}_R$ that

$$\vec{E}_{in}\ \mathrm{div}\ \vec{E}_R = \vec{E}_{in} \times \mathrm{rot}\ \vec{E}_R + \vec{E}_R \times \mathrm{rot}\ \vec{E}_{in} + \text{surface terms.}$$

Similarly the identity reduces for $\vec{A} = \vec{H}_{in}$ and $\vec{B} = \vec{H}_R$ to

$$\vec{0} = \vec{H}_{in} \times \mathrm{rot}\ \vec{H}_R + \vec{H}_R \times \mathrm{rot}\ \vec{H}_{in} + \text{surface terms.}$$

Hence, by combining these results with the Maxwell equations and assuming that the product of $\mathscr{F}_{(in)}$ and $\mathscr{F}_R$ vanishes sufficiently rapidly at spatial infinity, one arrives at the equation (50 d):

$$Q \int_{-\infty}^{t} dt [\vec{E}_{in}(\vec{x}(t)) + \vec{v} \times \vec{H}_{in}(\vec{\ }(t))]$$

$$= \frac{1}{4\pi} \int_{-\infty}^{t} dt \int dV \left\{ \vec{E}_{in} \times \mathrm{rot}\ \vec{E}_R + \vec{E}_R \times \mathrm{rot}\ \vec{E}_{in} + \vec{H}_R \times \mathrm{rot}\ \vec{E}_{in} + \vec{H}_{in} \times \frac{\partial \vec{E}_R}{\partial t} \right\}$$

$$= -\frac{1}{4\pi} \int_{-\infty}^{t} dt \int dV \left\{ \vec{E}_{in} \times \frac{\partial \vec{H}_R}{\partial t} + \vec{E}_R \times \frac{\partial \vec{H}_{in}}{\partial t} + \frac{\partial \vec{E}_{in}}{\partial t} \times \vec{H}_R + \frac{\partial \vec{E}_R}{\partial t} \times \vec{H}_{in} \right\}$$

$$= -\frac{1}{4\pi} \int dV \{ \vec{E}_{in} \times \vec{H}_R + \vec{E}_R \times \vec{H}_{in} \}.$$

Similarly, equation (49d) is immediately obtained by integration of the identity

$$\frac{2}{8\pi}\frac{\partial}{\partial t}\{\vec{E}_{in}\cdot\vec{E}_R+\vec{H}_{in}\cdot\vec{H}_R\} = \frac{1}{4\pi}\operatorname{div}(\vec{E}_{in}\times\vec{H}_R+\vec{E}_R\times\vec{H}_{in})-\vec{j}\cdot\vec{E}_{in}.$$

* * *

To evaluate the leading-order term in the energy and momentum associated with the domain Γ, the Heaviside ellipsoide — with semi-major axis ε — centred at the instantaneous position $\vec{x}(t)$ of the charge, is enclosed in the smallest possible lightsphere centred at the retarded position $\vec{x}(t_2)$. Since the terms proportional to $Q^2/2\varepsilon$ are independent of the acceleration, the motion may be assumed to be uniform between t_2 and t. Then, as is evident from the figure, the radius $R = t-t_2$ is related to ε and v by $R = \varepsilon\dfrac{\sqrt{1-v^2}}{1-v}$.

Furthermore the surface of the ellipsoide and the lightsphere is described by the relations

$$\left.\begin{aligned}\frac{1}{r_l} &= \frac{1}{\varepsilon\sqrt{1-v^2}}\sqrt{1-v^2\sin^2\vartheta}\\ \frac{1}{r_u} &= \frac{1}{R(1-v^2)}\left[v\cos\vartheta+\sqrt{1-v^2\sin^2\vartheta}\right],\end{aligned}\right\}$$

where r_l and r_u are defined on figure 5.

Since by assumption the field at time t between the two surfaces in question corresponds to that of a uniformly moving charge, the integral over the appropriate densities are easily evaluated to yield

$$\left.\begin{aligned}\Delta\mathscr{E}_{\Gamma-\Xi} &= \frac{1}{8\pi}\int_{\Gamma-\Xi} dV[\vec{E}^2+\vec{H}^2]_{\text{uniform}}\\ &= \frac{1}{8\pi}\int_{\Gamma-\Xi} dV\frac{Q^2}{r^4}\frac{(1-v^2)^2}{(1-v^2\sin^2\vartheta)^3}(1+v^2\sin^2\vartheta) = \frac{Q^2}{2\,\varepsilon}\frac{v}{1+v}\left(1+\frac{v^2}{3}\right)\gamma\\ \Delta\vec{\mathscr{P}}_{\Gamma-\Xi} &= \frac{1}{4\pi}\int_{\Gamma-\Xi} dV[\vec{E}\times\vec{H}]_{\text{uniform}}\\ &= \frac{\vec{v}}{4\pi}\int_{\Gamma-\Xi} dV\frac{Q^2}{r^4}\frac{(1-v^2)^2}{(1-v^2\sin^2\vartheta)^3}\sin^2\vartheta = \frac{Q^2}{2\,\varepsilon}\frac{\frac{4}{3}v}{1+v}\gamma\vec{v}.\end{aligned}\right\}$$

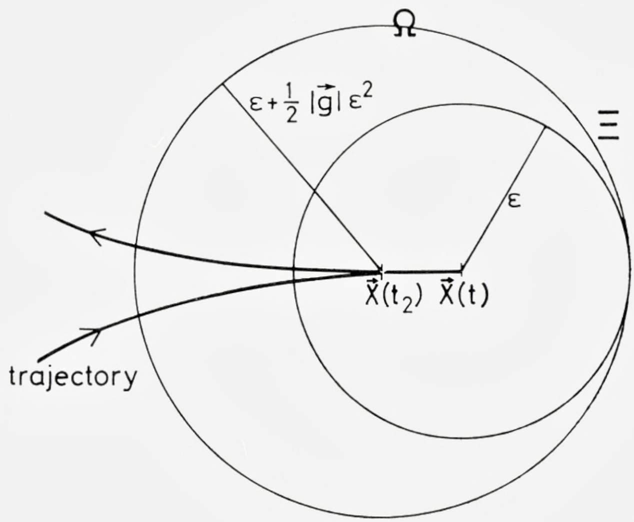

Fig. 6.

Hence, remembering that $t - t_2 = R$, one obtains the eq. (53) by means of the relation:

$$(\vec{\mathscr{P}}_\Gamma(t), \mathscr{E}_\Gamma(t) = (\vec{\mathscr{P}}_\Xi(t)), \vec{\mathscr{E}}_\Xi(t)) + (\varDelta\vec{\mathscr{P}}_{\Gamma-\Xi}, \varDelta\mathscr{E}_{\Gamma-\Xi}),$$

where $(\vec{\mathscr{P}}_\Xi(t), \mathscr{E}_\Xi(t))$ is given by eq. (51).

* * *

Finally, to evaluate the energy and momentum associated with the volume Ω, the sphere of radius ε centred at the instantaneous position $\vec{x}(t)$ is enclosed in the smallest possible lightsphere (i.e. of radius $t - t_2 \sim \varepsilon + \frac{1}{2}|\vec{g}|\varepsilon^2$, remembering $\vec{v}(t) = 0$) centred at the retarded position $\vec{x}(t_2)$ (see figure 6).

Since $|\vec{x}(t_2) - \vec{x}(t)| \sim \frac{1}{2}|\vec{g}|\varepsilon^2$, the volume $\Omega - \Xi$ between the two spheres is of the order of magnitude $|\vec{g}|\varepsilon^4$, and hence there is a finite amount of energy, $\varDelta\mathscr{E}_{\Omega-\Xi}$, associated with this volume even in the limit of vanishing ε. To the appropriate accuracy it is evident that (cf. the footnote on page 39)

$$\varDelta\mathscr{E}_{\Omega-\Xi} = \frac{1}{8\pi}\int\limits_{\Omega-\Xi}\frac{Q^2}{r^4}\,dV = \frac{Q^2}{\varepsilon^4}\int\frac{d\Omega}{8\pi}\,\varepsilon^2\tfrac{1}{2}|\vec{g}|\,\varepsilon^2(1-\cos\vartheta) = \frac{Q^2}{4}|\vec{g}|.$$

Whence, inserting $t - t_2 = \varepsilon + \frac{1}{2}|\vec{g}|\varepsilon^2$ into the expression for $\mathscr{E}_\Xi$ as given by eq. (51), one obtains the first of the eqs. (52): $\mathring{\mathscr{E}}_\Omega = \mathscr{E}_\Xi + \varDelta\mathscr{E}_{\Omega-\Xi}$. Since the momentum density associated with the volume in question is of order $Q^2|\vec{g}|/\varepsilon^3$, the corresponding momentum vanishes with ε. Thus the value of $\overset{\circ}{\vec{\mathscr{P}}}_\Omega$ may be immediately obtained by substituting $\vec{v}(t_2) \sim -\vec{g}(t)(t-t_2)$ into the expression (51) for $\vec{\mathscr{P}}_\Xi(t)$.

References

1) P. A. M. Dirac: Classical Theory of Radiating Electrons, Proc. Roy. Soc. (London) **A 167**, 148 (1938).
R. Haag: Die Selbstwechselwirkung des Elektrons, Z. Naturforschg. **10** a, 752, 1955.
2) R. P. Feynman and J. A. Wheeler: Interaction with the Absorber as the Mechanism of Radiation, Rev. Mod. Phys. **17**, 157 (1945).
Classical Electrodynamics in Terms of Direct Interparticle Action, Rev. Mod. Phys. **21**, 425 (1949).
3) F. Rohrlich: The Definition of Electromagnetic Radiation, Il Nuovo Cimento **XXI**, 811 (1961).
Classical Charged Particles (Addison-Wesley, Reading, Mass. 1965).
4) J. Kalckar and O. Ulfbeck: On the Problem of Gravitational Radiation, Mat. Fys. Medd. Dan. Vid. Selsk. **39** no. 6 (1974).
5) C. Møller: The Theory of Relativity, 2 ed. p. 458, (Clarendon Press, Oxford 1972).
6) C. Teitelboim: Splitting of the Maxwell Tensor: Radiation Reaction without Advanced Fields.
Phys. Rev. **D 1**, 1572 (1970).
Erratum, Phys. Rev. **D 2**, 1763 (1970).
Splitting of the Maxwell Tensor II. Sources, Phys. Rev. **D 3**, 297 (1971).
Radiation Reaction as a Retarded Self-Interaction, Phys. Rev. **D 4**, 345 (1971).

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